Stable Reduction of Algebraic Curves

Olivier de Gaay Fortman

August 2021

Stable Reduction

Olivier Gaay Fortman
August, 2021

1 Goal of this lecture

Let $\mathcal{M}_g$ be the moduli stack of smooth genus $g \geq 2$ curves. Goal of seminar: understand

$$H^*(\mathcal{M}_g, \mathbb{Q}).$$

Essential tool: Deligne-Mumford compactification

$$\mathcal{M}_g \subset \overline{\mathcal{M}}_g = \{\text{stable curves}\}.$$

After lunch, $\overline{\mathcal{M}}_g$ will be constructed. For the moment, assume it exists. Today we shall prove:

Theorem 1 (Deligne-Mumford, '69)**.** *$\overline{\mathcal{M}}_{g/\mathbb{C}}$ is proper. Equivalently, $\overline{M}_g(\mathbb{C})$ is compact Hausdorff.*

To see why these notions are equivalent, first observe that $\overline{M}_g(\mathbb{C})$ is compact Hausdorff if and only if $\overline{M}_g$ is proper over $\mathbb{C}$ [SGA1, Bourbaki]. So it suffices to show that $\overline{\mathcal{M}}_g$ is proper if and only if $\overline{M}_g$ is; since the morphism $\overline{\mathcal{M}}_g \to \overline{M}_g$ is finite, it suffices to prove that properness of $\overline{\mathcal{M}}_g$ implies properness of $\overline{M}_g$, but this follows from the Keel-Mori theorem: the coarse moduli space X of a separated DM stack $\mathcal{X}$ is a separated algebraic space, and hence proper if $\mathcal{X}$ is proper (Stacks).

2 Notation

- All schemes and stacks defined over $\mathbb{C}$.
- An algebraic variety is a reduced and separated scheme of finite type over $\mathbb{C}$.
- A curve (/surface) is a complete algebraic variety all whose irreducible components are of dimension one (/two).

3 Nodal and stable curves

Let $C = \cup_i C_i$ be a curve with normalization $\pi : \tilde{C} \to C$. Then $\mathcal{O}_C \to \pi_* \mathcal{O}_{C'}$ is injective. Indeed, let U be an affine open subset of C and let $A = \mathcal{O}_C(U)$. Let $\mathfrak{p}_i$ be the prime ideal corresponding to the generic point ξ_i of C_i. Then we have a finite canonical homomorphism

$$\varphi : A \to \oplus_i A/\mathfrak{p}_i.$$

Recall that $\sqrt{(0)} = \cap \mathfrak{p}$, the intersection of all prime ideals [Altman-Kleiman, 3.29], which is also the intersection of all minimal prime ideals of A [A-K, 3.14] - hence this intersection is zero since A is reduced. So φ is injective. It induces $\text{Frac}(A) \cong \oplus_i \text{Frac}(A/\mathfrak{p}_i)$. Since $V(\mathfrak{p}_i) = \text{Spec}(A/\mathfrak{p}_i)$ is an open subset of C_i, $V(\mathfrak{p}_i)$ is integral, hence $A/\mathfrak{p}_i$ is an integral domain. Moreover, the integral closure A' of A in $\text{Frac}(A)$ is contained in $\oplus(A/\mathfrak{p}_i)'$, where $(A/\mathfrak{p}_i)'$ is the integral closure of the domain $A/\mathfrak{p}_i$ in the field $\text{Frac}(A/\mathfrak{p}_i)$. It follows that $A' = \oplus(A/\mathfrak{p}_i)'$, and that $A \to A'$ is injective.

Define a coherent sheaf $\mathcal{S}$ on C by the following exact sequence:

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{C'} \to \mathcal{S} \to 0;$$

$\mathcal{S}$ is a skyscraper sheaf whose support is C_{sing}. If $\delta_x = \dim_{\mathbb{C}} \mathcal{S}_x$, then we obtain $n - \sum_i p_a(C_i') =$
$$\sum_i \chi(\mathcal{O}_{C_i'}) = \chi(\mathcal{O}_{C'}) = \chi(\pi_* \mathcal{O}_{C'}) = \chi(\mathcal{O}_C) + \chi(\mathcal{S}) = 1 - p_a(C) + \dim H^0(C, \mathcal{S}) = 1 - p_a(C) + \sum_x \delta_x.$$

Lemma 2. *Consider the curve $C = \cup_{i=1}^n C_i$ as above. Then*

$$p_a(C) + n - 1 = \sum_{i=1}^{n} p_a(C_i') + \sum_x \dim \delta_x.$$

□

Proposition 3. *Let $x \in C$. The following are equivalent:*

1. *$\pi^{-1}(x) = \{\alpha, \beta\}$ for some $\alpha, \beta \in \tilde{C}$ and $\delta_x = 1$,*

2. *We have an isomorphism*
$$\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[x,y]]/(xy).$$

3. *We have an isomorphism*
$$\hat{\mathcal{O}}_{X^{an},x} \cong \mathbb{C}[[x,y]]/(xy).$$

4. *Consider the analytic subset $X = \{xy = 0\} \subset \mathbb{C}^2$. There is an open neighborhood $x \in U \subset C^{an}$ and an open neighborhood $0 \in V \subset X$ such that $(U, x) \cong (V, 0)$.*

Proof. The direction 4 $\implies$ 3 is clear. For 3 $\iff$ 2, this follows from the fact that for any locally algebraic scheme X over $\mathbb{C}$, the morphism of ringed spaces X^{an} induces an isomorphism on completed local rings [SGA 1]. For 2 $\iff$ 1, see [Liu, 7.5.15]. We claim that 1 $\implies$ 4. By [Liu, proof of 10.3.7(d)], C is locally a closed (hence principal) subscheme of a smooth surface. If S is this surface, then S^{an} looks locally like $\mathbb{C}^2$, hence C is determined locally by a holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$. Then $f(0) = \partial f/\partial x(0) = \partial f/\partial y(0) = 0$, and that the Hessian of f at 0 is non-singular. Therefore 4 holds by [GACII, 10.2.3]. □

Definition 4. *Let C be a curve. A point $x \in C$ is a node if the above conditions are satisfied. A family of nodal curves is a proper flat morphism of schemes*

$$\varphi : \mathscr{X} \to B$$

such that every geometric fiber is a nodal curve. We also say that φ is a nodal S-curve.

Examples 5. *Draw pictures.*

Lemma 6. *Let $f : C \to S$ be a a nodal S-curve. Then f is a local complete intersection.*

Proof. Since f is flat and of finite type, it suffices to prove this in the case where S is the spectrum of an algebraically closed field k. We use the following *Lemma: Let $X \to S$ be a morphism of finite type over a locally Noetherian scheme S. Fix $s \in S$, $x \in X_s$, and let $d = \dim_{k(x)} \Omega^1_{X_s/k(s),x} \otimes_{\mathcal{O}_{X_s,x}} k(x)$. Then in a neighborhood of x, $X \to S$ factors into a closed immersion $X \to Z$ followed by a morphism $Z \to S$ which is smooth at x, and such that $\dim_x Z_s = d$ and that $\Omega^1_{Z/S,x}$ is free of rank d over $\mathcal{O}_{Z,x}$.* [Liu, 6.2.4]. Hence C is locally a closed - hence principal - subscheme of a smooth surface over k. Consequently, C is a local complete intersection over k. □

Corollary 7. *Let C be a nodal curve. Then C has a <u>canonical</u> sheaf ω_C [Liu, 6.4.7] which is isomorphic to the <u>dualizing</u> sheaf ω_C^o. In particular, the dualizing sheaf ω_C^o is invertible.*

Proof. See [Hartshorne, III.7.11]. □

Proposition 8. *Let C be a connected nodal curve of genus $g = p_a(C) \geq 2$. The following are equivalent:*

1. *Let E be a smooth rational irreducible component of C. Then E intersects the other components of C in more than 2 points.*

2. *$|\mathrm{Aut}(C)| < \infty$.*

3. ω_C *is ample.*

Proof. The equivalence of 1 and 2 is clear. Now let Q be the set of points of $\tilde{C}$ lying over nodes of C, and let $\{C_i\}$ be the irreducible components of C. For the equivalence of 2 and 3, one proceeds to show that, by the description of ω_C in terms of meromorphic differentials,

$$\deg(\omega|_{C_i}) = 2g(\tilde{C}_i) - 2 + |(Q \cap \tilde{C}_i)|. \tag{1}$$

But $\mathrm{Aut}(C)$ is finite if and only if the right side of (1) is larger than zero. Since a line bundle on a curve is ample if and only if its degree is positive on every irreducible component [ref: Liu], the result follows. ☐

Definition 9. *Let C be a curve. Then C is called stable if C is nodal, $g(C) = H^1(C, \mathcal{O}_C) \geq 2$, and the above conditions are satisfied. Let S be a scheme. A stable curve of genus g over S is a proper flat morphism $\pi : C \to S$ whose geometric fibers are stable curves of genus g.*

Examples 10. *Draw pictures.*

The following theorem is useful for constructing the moduli space of genus g stable curves:

Theorem 11 (Deligne-Mumford)**.** *Let C be a stable curve. Then $\omega_C^{\otimes 3}$ is very ample.*

4 Stable Reduction

Recall: our goal was to prove that $\overline{\mathcal{M}}_g$ (assuming it exists as a finite type Deligne-Mumford stack over $\mathbb{C}$) is proper over $\mathbb{C}$. The first step is separation. Why carry about stable curves? Let C be a smooth curve of genus $g \geq 2$ and let p be a point on C. Let $X = C \times C$ and let Y be the blow-up of X along (p, p). Then $\pi : Y \to C$ is a family of nodal curves of fiber $C_t = \pi^{-1}(t) = C$ whenever $t \neq p$, and C_p the union $C \cup_P \mathbb{P}^1$, the curve C glued to $\mathbb{P}^1$ at the point P. Both $X \to C$ and $Y \to C$ are families of nodal curves, extending the smmooth curve $X \setminus \{p, p\} \to C \setminus \{p\}$.

In other words, if we try to compactify $\mathcal{M}_g$ by throwing in all nodal curves, even if we manage to construct a moduli space, the result will not be separated. If we use stable curves, this does not happen:

Proposition 12. *Let X and Y be stable curves over a discrete valuation ring R with algebraically closed residue field. Denote by η and s the generic and closed points of* Spec R*, and assume that the generic fibres X_η and Y_η of X and Y are smooth. Then any isomorphism φ_η between X_η and Y_η extends to an isomorphism φ between X and Y.*

Proof. Start with a smooth curve X_η of genus $g \geq 2$ over the quotient field K of R, and let X be a stable curve over R with X_η as its generic fibre. Now given a smooth curve C of genus $g \geq 1$ over K, there is, up to canonical isomorphism, at most one regular 2-dimensional scheme Y, proper and flat over R, with C as its generic fibre, without exceptional curves of the first kind in Y_s. One can show that the existence of X implies the existence of a minimal model Y of X_η, and moreover that X is the normal scheme obtained from Y by contracting all non-singular rational components of Y_s linked to the other irreducible components by exactly two points. ☐

So what about stable curves?

Theorem 13 (Stable Reduction)**.** *Let $\mathcal{X} \to B$ be a proper flat curve over a smooth pointed curve $(B, 0)$ such that the restriction $\mathcal{X}^* \to B \setminus \{0\}$ is a stable genus $g \geq 2$ curve. There exists a finite cover $B' \to B$, totally ramified over 0, and a stable genus g curve $\widetilde{\mathcal{X}} \to B'$ over B' such that*

$$\widetilde{\mathcal{X}}|_{(B')^*} \cong \mathcal{X}^* \times_{B^*} (B')^*.$$

Diagram:

Instead of giving a complete proof, we first give an example and then give a sketch of the proof.

Example 14. *Consider a smooth projective surface S and an ample line bundle L on it, and let $\mathbb{P}^1 \subset \mathbb{P}H^0(S, L)$ be a projective line. This gives a pencil of curves $\{C_t\}$ on S; suppose that C_t is a smooth curve for t in a punctured neighborhood of $t = 0$, but that $C := C_0$ is a curve with one cusp $p \in C_0$. We can write the equation of the curve in a neighborhood of p and $t = 0$ as*

$$F(x, y) + t \cdot G(x, y) = 0,$$

with G nonzero at p. Locally, such a pencil will look like $y^2 = x^3 + t$. Let $\tilde{C} \to C$ be the normalization of C. Then the stable limit is $\tilde{C} \cup_p E$, the curve $\tilde{C}$ with an elliptic tail at $p \in \tilde{C}$.

Proof. Let $\mathscr{X} \subset \mathbb{P}^1 \times S$ be the total space of the family, and write the family by $\pi : \mathscr{X} \to B$. Notice that $\mathscr{X}$ is smooth. We have $C = X_0$, an effective irreducible divisor on $\mathscr{X}$. First blow up the point $p \in C$: write $\varphi_1 : Bl_p\mathscr{X} = \mathscr{X}_1 \to \mathscr{X}$.

Draw picture.

This amounts to replacing the divisor C by $\varphi^*C = \widetilde{C} + 2E_1$, where $E_1 \cong \mathbb{P}^1$. Note that

$$\widetilde{C} \cdot E_1 = (\varphi^*C - 2E_1) \cdot E_1 = -2E_1^2 = 2.$$

If $\widetilde{C} \cap E_1 = \{p_1, p_2\}$, then there are two points of $\widetilde{C}$ lying over the node $p \in C$; this is absurd, hence $\widetilde{C}$ and E_1 intersect in a single point $p \in \mathscr{X}_1$, and have intersection multiplicity 2 there.

Next, we blow up the point $p \in \mathscr{X}_1$ to get $\mathscr{X}_2$: write $\varphi : \mathscr{X}_2 \to \mathscr{X}_1$.

Draw picture.

This creates an extra smooth rational curve $E_2 \subset \mathscr{X}_2$. Note that $\varphi^*(\widetilde{C} + 2E_1) = \widetilde{C} + 2E_1 + 3E_1$ as divisors on $\mathscr{X}_2$. One observes that

$$\widetilde{C} \cdot E_1 = (\widetilde{C} - E_2) \cdot (E_1 - E_2) = \widetilde{C} \cdot E_1 + E_2^2 = 2 - 1 = 1.$$

Similarly, $\widetilde{C} \cdot E_2 = E_1 \cdot E_2 = 1$. Suppose that $\widetilde{C}$ meets E_1 and E_2 in different points. Since $\widetilde{C}$ and E_1 meet transversally now, and their intersectio number on $\mathscr{X}_2$ outside E_2 is the same as their intersection number on $\mathscr{X}_1$, so this is absurd. Hence $\widetilde{C}$ intersects E_1 and E_2 in the same point $p \in \mathscr{X}_2$. We have:

$$\varphi_2^*(\widetilde{C} + 2E_1) = \widetilde{C} + 2E_1 + 3E_2.$$

Next, we blow up $p \in \mathscr{X}_2$: write $\varphi_3 : \mathscr{X}_3 \to \mathscr{X}_2$ for this morphism.

Draw picture.

Note that $\widetilde{C} \cdot E_1 = (\varphi^*\widetilde{C} - E_3) \cdot (\varphi^*E_1 - E_3) = \widetilde{C} \cdot E_1 + E_3^2 = 0$, and similarly $\widetilde{C} \cdot E_2 = E_1 \cdot E_2 = 0$. Moreover, we have

$$\varphi_3^*(\widetilde{C} + 2E_1 + 3E_2) = \widetilde{C} + 2E_1 + 3E_2 + 6E_3.$$

We have thus arrived at a family whose reduced special fiber has only nodes as singularities; but the special fiber is non-reduced, have components of multiplicity 2, 3 and 6.

Definition 15. *For any divisor $D = \sum a_i D_i$ on a surface, and $p \in \mathbb{Z}$, define $D_{\equiv p}$ to be the divisor $D_{\equiv p} = \sum \bar{a}_i D_i$ where $0 \leq a_i \leq p - 1$ and $\bar{a}_i \equiv a_i \mod p$.*

Lemma 16. *Consider our family $\mathscr{X} \to B$ above, with special fiber $X_0 = D = \sum a_i D_i$. For any prime number p, let $\widetilde{\mathscr{X}}$ be the normalization of the base change of $\mathscr{X} \to B$ along $B \to B, t \mapsto t^p$. Then $\widetilde{\mathscr{X}} \to \mathscr{X}$ is a finite cover whose ramification divisor is $D_{\equiv p} \subset \mathscr{X}$. Moreover, $\widetilde{\mathscr{X}}$ is smooth if $D_{\equiv p}$ is smooth.*

Proof. Let $\mathscr{X}' \to B$ be the base change of $\mathscr{X} \to B$ along $B \to B$, $t \mapsto t^p$. Then $\mathscr{X}' \to \mathscr{X}$ is a degree p cover ramified along $D = \{t = 0\}$, so that the local equation of the surface $\mathscr{X}'$ is $\{(u, x) \in B \times \mathscr{X} : u^2 = \pi(x) = t\}$ everywhere. Let $E \subset \mathscr{X}$ be a component of multiplicity $m = a + pk$, $0 \le a \le p-1$ in the special fiber. Let $p \in D$ and let $g \in \mathfrak{m}_p \subset \mathcal{O}_{\mathscr{X},p}$ be the Cartier divisor defining D_{red} around p. Then in a neighborhood of p, $t = g^m$, so that the local equation of $\mathscr{X}'$ will be

$$\{(u, x) : u^p = g^m(x)\}.$$

If $m > 1$, this will be singular along the inverse image of D. The normalization process will replace u by a local coordinate $v = u/g^{[m/p]} = u/g^k$, so that the local equation of the normalization will be

$$v^p = \frac{u^p}{g^{kp}} = \frac{g^m}{g^{kp}} = g^a.$$

So indeed, $\widetilde{\mathscr{X}} \to \mathscr{X}$ is a ramified degree p cover, whose ramification divisor is $D_{\equiv p}$. □

In our case, $D = (t) \subset \mathscr{X}$ reduced mod 2 is $D_{\equiv 2} = \widetilde{C} + E_2$. Since $D_{\equiv 2}$ is smooth, $\widetilde{X}$ will be smooth as well. The inverse images of E_2 and $\widetilde{C}$ will be curves mapping isomorphically to them. Since E_3 meets the branch locus in two points, its inverse image Y will be a smooth double cover of $E_3 \cong \mathbb{P}^1$ ramified at two points: by Riemann-Hurwitz, this gives $2g(Y) - 2 = 2(-2) + 2 = -2 \implies g = 0$: Y is a single rational curve and we write $E_3 = Y \subset \widetilde{\mathscr{X}}$. Since E_1 is disjoint from the branch locus, $\varphi^{-1}E_1$ is an unramified cover of $E_1 \cong \mathbb{P}^1$: two disjoint smooth rational curves which we call E_1' and E_1''. The pullback to $\widetilde{\mathscr{X}}$ of the divisor $D = (t)$ on $\mathscr{X}$ is the sum of the components of the inverse image of the special fiber in $\mathscr{X}$, which multiplicities unchanged from that of the corresponding component of (t) on $\mathscr{X}$ for those components that are not contained in the branch divisor, and which multiplicity doubled for components in the branch divisor. Therefore, we have:

$$\varphi^* D = 2\widetilde{C} + 2E_1' + 2E_1'' + 6E_2 + 6E_3.$$

The special fiber (u) of the new family $\widetilde{\mathscr{X}} \to B$ is exactly one-half of this divisor: thus

$$(u) = \widetilde{X}_0 = \widetilde{C} + E_1' + E_1'' + 3E_2 + 3E_3.$$

–

Draw picture.

Write $\mathscr{X} \to B$ for the new family, with special fiber $D = (u)$ as given above. Make a base change of order 3. Note that p^{th} order covers of $\mathbb{P}^1$ totally ramified along two points have genus g determined by

$$2g - 2 = -2p + 2(p-1);$$

that is, $g = 0$. Note that $D_{\equiv 3} = \widetilde{C} + E_1' + E_1''$. Write $\varphi : \widetilde{\mathscr{X}} \to \mathscr{X}$ for the 3rd order cover ramified along $D_{\equiv 3}$ as above. The inverse images of $\widetilde{C}?E_1$ and E_1'' are copies of themselves. Since E_2 is disjoint from $D_{\equiv 3}$, its inverse image is a disjoint union of three smooth rational curves, which we call E_2', E_2'' and E_2'''. The inverse image E of E_3 is a smooth triple cover of $E_3 \cong \mathbb{P}^1$ totally ramified over three points. Riemann-Hurwitz gives

$$2g(E) - 2 = -6 + 3 \cdot 2 \implies g(E) = 1.$$

In other words, E is an elliptic curve. We have

$$\varphi^* D = 3\widetilde{C} + 3E_1' + 3E_1'' + 3E_2' + 3E_2'' + 3E_2''' + 3E.$$

Let v be the local coordinate on B such that $v^3 = u$. Write $\pi : \mathscr{X} \to B$ as the new family thus obtained. Then

$$(v) = X_0 = \widetilde{C} + E_1' + E_1'' + E_2' + E_2'' + E_2''' + E.$$

–

Draw picture.

Then $\pi : \mathscr{X} \to B$ is a family whose special fiber X_0 is a reduced curve with only nodes as singularities. Note that for any component F in the special fiber X_0 we have $F \cdot X_0 = 0$, since in fact X_0 is a principal Cartier divisor, defined by the meromorphic function $v : \mathscr{X} \to \mathbb{P}^1$, $v \in \mathbb{C}(\mathscr{X})$. It follows that for any rational curve F in X_0, we have

$$F \cdot X_0 = F^2 + F \cdot E = F^2 + 1 = 0 \implies F^2 = -1.$$

Hence F is an exceptional curve of the first kind and can be contracted by Castelnuovo's theorem [Hartshorne, V.5.7]. Blowing down the five curves of this tape, we arrive at a family

$$\pi : \mathscr{X} \to B$$

whose special fiber consists of the union of the normalization $\widetilde{C}$ of the original curve together with the elliptic curve E (called an *elliptic tail*), joined at the point of $\widetilde{C}$ lying over the cusp of C.

Draw picture.

Then $\pi : \mathscr{X} \to B$ is the stable reduction. (Note that E is the unique elliptic curve with an automorphism of order 3. Its j-invariant is $j(E) = 0$.) □

5 Proof of Stable Reduction

Sketch of Proof of Theorem 13. –

I. We may assume that our family $\pi : \mathscr{X} \to B$ is smooth over $B \setminus \{0\}$. Indeed, this follows from the fact that we have already proved that $\overline{\mathcal{M}}_g$ is separated (see Proposition 12) so this follows from [Stacks, Lemma 0CQM].

II. Apply resolution of singularities to the pair $(\mathscr{X}, X_0)$: thus we may assume that $\mathscr{X}$ is smooth, and that $(X_0)_{red}$ is a normal crossings divisor. At this point, the map π will by given by an equation of the form $t = x^a y^b$ in terms of a local coordinate t on B and local coordinates x and y on $\mathscr{X}$.

III. Let m be the least common multiple of the multiplicities of the components of the special fiber X_0. Make a base change $t \mapsto t^m$ and normalize the resulting total space. A local calculation then shows that X_0 has reduced normal crossings and the map π has local equation of the form either $t^n = x$ or, at nodes of the special fiber, $t^n = xy$ where t is again a local coordinate on B. In the latter case, the total space $\mathscr{X}$ will be smooth at the node if and only if $n = 1$. If $n > 1$, there is an A_{n-1} singularity at the node. In any case, X_0 is now reduced and nodal.

IV. Minimally resolve the A_{n-1} singularities that arise. This has the effect of replacing each singularity by a chain of $(n-1)$ smooth rational curves. Now we have a family $\mathscr{X} \to B$ with smooth total space and reduced, nodal special fiber.

V. Blow down all exceptional curves of the first kind: these are smooth rational components of X_0 meeting the rest of X_0 only once. This gives the minimal model $\mathscr{X} \to B$ of $\mathscr{X}$: given any smooth curve $Y \to B \setminus \{0\}$, there is, up to canonical isomorphism, at most one regular surface $\bar{Y}$ together with a flat and proper morphism $\bar{Y} \to B$, restricting to Y over $B \setminus \{0\}$, without exceptional curves of the first kind.

VI. To obtain stable reduction, blow down all semistable chains of smooth rational curves: that is, chains of smooth rational curves of self-intersection -2.

□

6 Stable Reduction in all characteristics

There is in fact a stronger version of Theorem 13:

Theorem 17 (Deligne-Mumford, '69)**.** *Let R be a discrete valuation ring with fraction field K. Let η and s be the generic and closed point of* $\mathrm{Spec}(R)$ *respectively. Let C be a smooth geometrically irreducible curve over K of genus $g \geq 2$. There exists a finite algebraic extension L of K and a stable curve $\mathscr{C}_L \to \mathrm{Spec}(R_L)$, where R_L is the integral closure of R in L, such that $\mathscr{C}_{L,\eta} \cong C \times_K L$.*

Sketch of the proof. Let $\mathscr{C}$ be the minimal model of C over R: $\mathscr{C}$ is to be a regular scheme, flat and proper over R, with generic fiber $\mathscr{C}_\eta = C$, such that for any other regular scheme $\mathscr{C}'$, flat over R with generic fiber $\mathscr{C}'_\eta = C$, the birational map $\mathscr{C}' \to \mathscr{C}$ is a morphism. This scheme exists [Safarevich, Lichtenbaum] and is clearly unique for these properties. Moreover, $\mathscr{C}$ is projective over R [Lichtenbaum, see Liu, 8.3.16].

Let A be an abelian variety over K. Let $\mathcal{A}^0$ be the identy component of the Néron model of A over R. We say that A has *semi-abelian reduction over R* if $\mathcal{A}^0_s$ is a semi-abelian variety. That is, there is an exact sequence of algebraic groups

$$0 \to T \to \mathcal{A}^0_s \to B \to 0$$

where T is a torus and B an abelian variety over $k(s)$. Moreover, we say that C has stable reduction in *sense 1* if $\mathscr{C}_s$ is a nodal curve. We say that C has stable reduction in *sense 2* if there is a stable curve $\mathscr{X}$ over R with generic fibre $\mathscr{X}_\eta = C$.

Proposition 18. *The two senses of stable reduction for C are equivalent.*

Proof. See [DM, 2.3]. □

Let J be the Jacobian of C. It is shown in [DM, Theorem 2.4] that J has stable reduction if and only if C has stable reduction. Moreover, there is the following

Theorem 19 (Grothendieck)**.** *Let R be a discrete valuation ring with quotient field K. Let A be an abelian variety over K. Then there exists a finite algebraic extension L of K such that, if R_L is the integral closure of R in L, then $A \times_K L$ has semi-abelian reduction over R.*

This concludes the proof. □

Bibliography

[1]

[2]

[3]