DEFORMATIONS OF SINGULARITIES

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1. Introduction

This note is written with the goal of presenting some interesting and important results in the theory of deformations of singularities. The references we use are [Har10] and [Sta18].

- We define the Schlessinger-Lichtenbaum complex [LS67] and show that it coincides with the truncation of the cotangent complex at the -2 level (Theorem 4.1).
- Using the cotangent complex of a morphism of rings, one defines T^i functors for any $i \in \mathbb{Z}_{\geqslant 0}$ (Definition 5.1); these extend T^0, T^1, T^2 as defined in [LS67] because of (1).
- Recall the following:

Theorem 1.1. *[EGA, IV, Ch. 0, §20] Let $A \to B$ and $B \to C$ be morphisms of rings, and let M be a C-module. There is a canonical exact sequence of A-modules*

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \to \mathrm{Exal}_B(C, M) \to \mathrm{Exal}_A(C, M) \to \mathrm{Exal}_A(B, M)$$

which is functorial in M.

Now for any morphism of rings $A \to B$ and any B-module M, one has $T^0(B/A, M) = \mathrm{Der}_A(B, M)$ and $T^1(B/A, M) = \mathrm{Exal}_A(B, M)$ (Proposition 5.2). Moreover, the exact sequence of Theorem 1.1 extends to an infinite exact sequence of A-modules

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \to \mathrm{Exal}_B(C, M) \to \mathrm{Exal}_A(C, M) \to \mathrm{Exal}_A(B, M) \to$$
$$\to T^2(C/B, M) \to T^2(C/A, M) \to T^2(B/A, M) \to T^3(C/B, M) \to T^3(C/A, M) \to T^3(B/A, M) \to \dots$$

- Similarly, for a ring map $A \to B$, any short exact sequence of B-modules $M' \to M \to M''$ induces an infinite long exact sequence of B-modules (Theorem 5.3)

$$\dots \to T^{i-1}(B/A, M'') \to T^i(B/A, M') \to T^i(B/A, M) \to T^i(B/A, M'') \to T^{i+1}(B/A, M') \dots$$

- We recall the definition of the cotangent complex of a morphism of schemes in Section 6 and present the result that a flat morphism of finite type between noetherian schemes is smooth if and only if $\mathcal{T}^1$ vanishes on coherent sheaves, and a relative local complete intersection if and only if $\mathcal{T}^2$ vanishes on coherent sheaves (Theorem 6.1).
- We prove that deformations of an affine scheme $X = \mathrm{Spec}\, B$ over a field k are parametrised by $T^1(B/k, B)$ (Theorem 7.1). If X is finite over k then the dimension of this deformation space is finite.
- We apply the theory to calculate the deformation space of the cusp, the ordinary double point on a surface and the cone over the Veronese surface (Examples 7.2).

2. Summary of previous results

Notation 1. In this note, k is a field and $D = k[t]/(t^2) = k[\epsilon]$ is the algebra of dual numbers.

Let us recall what we have seen so far.

(1) In Wouter's talk we have seen deformation of subschemes and coherent sheaves: if $Y \subseteq X$ is a closed subscheme of a scheme X over k, then
$$\mathrm{Def}_{Y/X}(D) \cong H^0(Y, \mathcal{N}_{Y/X}),$$
and $\mathcal{F}$ a coherent sheaf on X, then
$$\mathrm{Def}_{\mathcal{F}}(D) \cong \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}).$$

(2) Dirk showed that if k is algebraically closed, and X a nonsingular algebraic variety over k, then
$$\mathrm{Def}_X(D) \cong H^1(X, \mathcal{T}_X).$$
In particular, nonsingular affine algebraic varieties over k are rigid.

(3) Still assume k to be algebraically closed. Renjie proved under what conditions deformations over local artin algebras lift: suppose that $0 \to I \to B \to A \to 0$ is a small extension in Art_k, X a nonsingular algebraic variety over k, $\xi = (\mathcal{X}, i)$ a deformation of X over A. There exists a function
$$\mathcal{O}_-(e) : \mathrm{Def}_X(A) \to H^2(X, \mathcal{T}_X \otimes I)$$
such that $\mathcal{O}_\xi(e) = 0$ if and only if the deformation ξ of X to A over k lifts to a deformation of ξ to B over A. If $\mathcal{O}_\xi(e) = 0$ then the set of liftings is a torsor under $H^1(X, \mathcal{T}_X \otimes I)$. This generalises (2). Renjie also proved that a quotient ring R of a regular local ring P with kernel $J \subset M_P^2$ admits a tangent-obstruction (T-O) theory, and that the local Hilbert functor and the deformation functor of the projective line are pro-representable.

(4) Let k be algebraically closed. Mike showed that a deformation functor $F : \mathrm{Art}_k :\to \mathrm{Set}$ is pro-representable if and only if it satisfies Schlessinger's criteria.

(5) Let k be algebraically closed. Kees proved a couple of nice results: first of all, any deformation functor $F : \mathrm{Art}_k \to \mathrm{Set}$ with T-O theory satisfies Schlessinger's Criterion. Next, he generalised Wouter's result on deformations on closed subschemes: consider a small extension $0 \to I \to B \to A \to 0$, closed subscheme $Y_0 \subset X_{0/k}$, $Y \subset X_{/A}$ deformation of $Y_0 \subset X_0$ over k, $X'_{/B}$ a deformation of X over A. Then $\mathrm{Def}_{Y/X,X'}(B/A)$ is a pseudo-torsor under $H^0(Y_0, \mathcal{N}_{Y_0/X_0} \otimes I)$, and if Y' exists locally, then there exists an element $\alpha \in H^1(Y_0, \mathcal{N} \otimes I)$ such that $\alpha = 0$ if and only if Y' exists globally. This generalises (1). We also saw that the embedded deformation functor H_{Y_0/X_0} has T-0 theory with $T_i = H^{i-1}(Y_0, \mathcal{N}_{Y_0/X_0})$. Smooth schemes admit T-O theory: if $X_0 \to \mathrm{Spec}\, k$ is smooth, then Def_{X_0} has T-O theory with $T_i = H^i(X_0, T_{X_0})$. Kees showed that a natural transformation F between deformation functors with T-O theory induces morphisms between tangent and obstruction spaces. These morphisms are surjective resp. injective if and only if F is smooth. The forgetful functor $H_{Y_0/X_0} \to \mathrm{Def}_{Y_0}$ is smooth if X_0 and Y_0 are smooth.

(6) Emelie outlined some results on deformations of morphisms. For $i \in \{1,2\}$, let $(\mathbb{A}_i : 0 \to I_i \to A_i' \to A_i \to 0) \in \mathrm{Exal}_{\mathbb{Z}}(A_i, I_i)$. For each i, consider $(\mathbb{B}_i : 0 \to N_i \to B_i' \to B_i \to 0) \in \mathrm{Exal}_{\mathbb{Z}}(B_i, N_i))$ and let $\mathbb{A}_1 \to \mathbb{A}_2$, $\mathbb{A}_i \to \mathbb{B}_i$, $i \in \{1,2\}$ be morphisms of exact sequences. Consider morphisms $N_1 \to N_2$ and $B_1 \to B_2$ making everything commute. There exists a canonically defined element $\mathcal{O}(B_1', B_2') \in \mathrm{Ext}^1_{B_1}(\mathrm{NL}_{B_1/A_1}, N_2)$ such that $\mathcal{O}(B_1', B_2') = 0$ if and only if there exists a morphism $B_1' \to B_2'$ making everything commute. The set of all $B_1' \to B_2'$ as is a pseudo-torsor under $\mathrm{Hom}_{B_1}(\Omega_{B_1/A_1}, N_2)$.

(7) Without defining the cotangent complex of a morphism of rings $A \to B$, Maciek showed what properties such a functor $\mathbb{L}_{-/A} : \mathrm{Alg}_A \to D(A)$ with $\mathbb{L}_{B/A} \in D(B)$ for all $B \in \mathrm{Alg}_A$ is supposed to have. Let B_0/k be an algebra and $0 \to I \to A' \to A \to 0$ a small extension in Art_k. Assume B/A is flat and $B \otimes_A k = B_0$. Then there exists a class $\eta_B \in \mathrm{Ext}^2(L_{B_0/k}, B_0) \otimes_k I$ whose vanishing is necessary and sufficient for the existence of a lifting B'/A' satisfying $B' \otimes_{A'} A \cong B$. Moreover, the set of such liftings is a pseudo-torsor under $\mathrm{Ext}^1(L_{B_0/k}, B_0) \otimes_k I$. Then Maciek proved that for any perfect algebra k over $\mathbb{F}_p$ and any $n \geqslant 1$, the $\mathbb{Z}/p^n$-algebra of Witt vectors $W_n(k)$ exists. Moreover, for any perfect algebra k over $\mathbb{F}_p$, $L_{\mathbb{F}_p/k} = 0$. Some other properties of the cotangent complex: $L_{S^{-1}A/A} = 0$ for multiplicative subsets $S \subset A$; if $A \to B$ smooth then $L_{B/A} = \Omega^1_{B/A}[0]$ (hence zero for $A \to B$ étale); $L_{B/A}$ commutes with flat base change; if $A \to B$ surjective with kernel generated by a regular sequence then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$; for a local complete intersection $A \to B$, $\mathbb{L}_{B/A}$ is a perfect complex supported in degrees $[-1, 0]$. Then Maciek lifted deformations of quotients by groups: if $X \to Y$ is the geometric quotient of scheme X by a free action of an abstract group G, then for any deformation $\tilde{Y}$ of Y there exists a deformation $\tilde{X}$ of X and a free group action on $\tilde{X}$ such that $\tilde{X}/G = \tilde{Y}$. Finally, Maciek gave a condition to be a local complete intersection morphism: for a morphism of rings $R \to S$, the following is true: if S has a finite resolution by flat R-modules and the cotangent complex $L_{S/R}$ is quasi-isomorphic to a bounded complex of flat S-modules, then $R \to S$ is a local complete intersection.

(8) Lenny gave a construction of the cotangent complex. Before doing so, he recalled some of the properties of the cotangent complex of a ring map $R \to A$: one has $H^0(\mathbb{L}_{A/R}) = \Omega^1_{A/R}$; if A/R is smooth, then $\mathbb{L}_{A/R} = \Omega^1_{A/R}[0]$; $\mathrm{Hom}_A(\mathbb{L}_{A/R}, M) = \mathrm{Der}_R(A, M)$; one has $\mathrm{Ext}^1_A(\mathbb{L}_{A/R}, M) = \mathrm{Exal}_R(A, M)$; for $R \to A \to B$ ring maps, we have a distinguished triangle

$$\mathbb{L}_{A/R} \otimes^{\mathbf{L}}_A B \to \mathbb{L}_{B/R} \to \mathbb{L}_{B/A} \to \mathbb{L}_{A/R} \otimes^{\mathbf{L}}_A B[1]$$

in $D^{\leqslant 0}(B)$; we have $\tau_{\geqslant -1}\mathbb{L}_{A/R} = \mathrm{NL}_{A/R}$, the naive cotangent complex; and $\mathbb{L}_{A/R}$ can be computed using a smooth resolution. The proof of existence goes as follows. It can be shown that for any R-algebra A, a free simplicial resolution $P_\bullet \to A$ exists in Mod_R - in fact, there exists a *canonical* free simplicial resolution $P_\bullet \to A$. This defines a functor

$$\mathrm{Alg}_R \to \mathrm{hoSimp}(\mathrm{Alg}_R), A \mapsto P_\bullet.$$

We can then simply define the functor $\mathbb{L}_{-/R}$ to be the composition

$$\text{(1)}\quad \mathrm{Alg}_R \xrightarrow{A\mapsto P_\bullet} \mathrm{hoSimp}(\mathrm{Alg}_R) \xrightarrow{\Omega_{-/R}} \mathrm{hoSimp}(\mathrm{Mod}_R) \xrightarrow{\text{Dold-Kan}} \mathrm{hoCh}_{\leqslant 0}(\mathrm{Mod}_R) \to D_{\geqslant 0}(R)$$

and observe that, for any A in Alg_R, $\mathbb{L}_{A/R}$ lands in $D_{\geqslant 0}(A)$ because $\Omega_{P_\bullet/R}$ is a $P_\bullet$-module and $P_\bullet \to A$ a quasi-isomorphism.

Our interest is the deformation of singularities, and for this we will need some of the above results. Let us rephrase them as follows.

Theorem 2.1. *For a ring A, there is a functor $\mathbb{L}_{-/A} : \mathrm{Alg}_A \to D_{\geqslant 0}(A)$ such that $\mathbb{L}_{B/A} \in \mathrm{Ob}(D_{\geqslant 0}(B))$ for any A-algebra B, and such that moreover*

1. *if A/R is smooth, then $\mathbb{L}_{A/R} = \Omega^1_{A/R}[0]$,*
2. *in general, $H^0(\mathbb{L}_{A/R}) = \Omega^1_{A/R}$, which implies that*
3. $\mathrm{Hom}_A(\mathbb{L}_{A/R}, M) = \mathrm{Der}_R(A, M)$,
4. $\mathrm{Ext}^1_A(\mathbb{L}_{A/R}, M) = \mathrm{Exal}_R(A, M)$,
5. *for $R \to A \to B$ ring maps, we have a distinguished triangle*
$$\mathbb{L}_{A/R} \otimes^{\boldsymbol{L}}_A B \to \mathbb{L}_{B/R} \to \mathbb{L}_{B/A} \to \mathbb{L}_{A/R} \otimes^{\boldsymbol{L}}_A B[1]$$
in $D^{\leqslant 0}(B)$,
6. $\tau_{\geqslant -1}\mathbb{L}_{A/R} = \mathrm{NL}_{A/R}$.

Remark 2.2. Recall that, for any A-algebra B, a *free simplicial resolution* $P_\bullet \to B$ is a simplicial object $P_\bullet \in \mathrm{Ob}\mathrm{Simp}(\mathrm{Alg}_R)$ with each P_i a free polynomial R-algebra together with an augmentation map $P_0 \to B$ of A-algebras such that

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

is exact in Mod_R. There is a canonical free simplicial resolution $P_\bullet \to B$ giving $\mathrm{Alg}_A \to \mathrm{Simp}(\mathrm{Alg}_A)$ and so we can define the *cotangent complex* $\mathbb{L}_{B/A}$ of the ring map $A \to B$ as an actual cochain complex $\mathbb{L}_{B/A}$ of B-modules (and not just its image in $D^{\leqslant 0}(B)$).

Remark 2.3. In $D(B)$, we have the identification

$$\mathbb{L}_{B/A} = \mathrm{Comp}(\Omega_{P_\bullet/A} \otimes_{P_\bullet} B) \qquad \in \mathrm{Ob}\mathrm{Comp}(\mathrm{Mod}_B)$$

where $\mathrm{Comp}(M_\bullet)$ means taking the cochain complex attached to a simplicial B-module $M_\bullet$. In the sequel we shall just write $\Omega_{P_\bullet/A} \otimes_{P_\bullet} B$ when we mean the complex associated to the simplicial B-module $\Omega_{P_\bullet/A} \otimes_{P_\bullet} B$.

Remark 2.4. For any free resolution $P'_\bullet \to B$ we have a canonical isomorphism

$$\mathbb{L}_{B/A} = \Omega_{P'_\bullet/A} \otimes_{P'_\bullet} B$$

in $D(B)$ [Sta18, Tag 08QI].

3. The Schlessinger-Lichtenbaum Complex

Let $A \to B$ be a morphism of rings. In [LS67] there is an explicit determination of $\tau_{\geqslant -2}\mathbb{L}_{B/A}$ which is used in calculations of versal deformation spaces of singularities. The construction is as follows.

Choose a polynomial ring $P = A[X]$ on a set X such that B is the quotient of R as an A-algebra. Let I be the ideal defining B, choose generators f_t for I indexed by a set T so that there is a free P-module $F = \oplus_{t \in T} P$ and a surjection $j : F \to I$ mapping e_t to f_t. Let $Q \subset F$ be the kernel of j. Let $F_0 \subset Q$ be the submodule of relations of the form $j(a)b - j(b)a$ with $a, b \in F$. Define

$$L_2 := Q/F_0, L_1 := F \otimes_P B = F/IF, L_0 := \Omega_{P/A} \otimes_P B$$

and the maps between them be defined as in the following diagram, where all rows are exact,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F_0 & \longrightarrow & Q & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & IF & \longrightarrow & F & \longrightarrow & L_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & \searrow & \\
 & & & & \downarrow & & I/I^2 & \longrightarrow & \Omega_{R/A} \otimes_R B = L_0 \\
 & & & & \downarrow & & & & \\
 & & 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & B \longrightarrow 0. \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Lemma 3.1. *Up to canonical isomorphism, the object $L = L_{R,F} \in \mathrm{Ob}D(B)$ attached to the complex $L_\bullet = (Q/F_0 \to F \otimes_P B \to \Omega_{P/A} \otimes_P B)$ does not depend on the choice of P and F.*

Proof. Either by direct calculation [Har10, Lemma's 3.2 & 3.3] - i.e. fix P and consider $F \twoheadrightarrow I, F' \twoheadrightarrow I$, take $F \oplus F'$, change its basis and show that $L_{P,F\oplus F'}$ and $L_{P,F}$ differ by a direct summand with a free complex hence $L_{P,F} \in D(B)$ does not depend on F, then do something similar for $P \twoheadrightarrow B, P' \twoheadrightarrow B$ - or use Theorem 4.1 below! $\square$

4. Comparison with the Cotangent Complex

Theorem 4.1. *There is a canonical map*

$$\mathbb{L}_{B/A} \to L$$

in $D(A)$ which induces an isomorphism $\tau_{\geqslant -2}\mathbb{L}_{B/A} \xrightarrow{\sim} L$ in $D(B)$.

Proof. Let $P_\bullet \to B$ be a free simplicial resolution of B over A. Identify $\mathbb{L}_{B/A}$ with $\Omega_{P_\bullet} \otimes_{P_\bullet} B$ (see Remark 2.2). Our aim is to define morphisms

$$\Omega_{P_0/A} \otimes_{P_0} B \to \Omega_{P/A} \otimes_P B, \quad \Omega_{P_1/A} \otimes_{P_1} B \to F \otimes_P B, \quad \Omega_{P_2/A} \otimes_{P_2} B \to Q/F_0$$

that make Diagram (3) commute and check that the morphisms $H_0(\mathbb{L}_{B/A}) \to H_0(L)$, $H_1(\mathbb{L}_{B/A}) \to H_1(L)$ and $H_2(\mathbb{L}_{B/A}) \to H_2(L)$ which are induces by the so obtained morphism of complexes $\mathbb{L}_{B/A} \to L$ are isomorphisms.

<u>Step 1: Biderivations</u>

Definition 4.2. Let $A \to B$ be a ring map. Let M be a (B, B)-bimodule over A. An A-biderivation is an A-linear map $\lambda : B \to M$ such that $\lambda(xy) = x\lambda(y) + \lambda(x)y$.

Lemma 4.3. *Let $P = A[S]$ be a polynomial ring over A. Let M be a (P, P)-bimodule over A. Then the function*

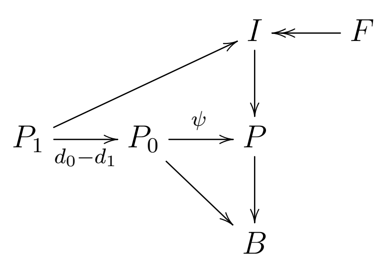
$$\mathrm{BiDer}_A(P, M) \to \mathrm{Hom}_{\mathrm{Set}}(S, M), \quad \lambda \mapsto \lambda|_S$$

is bijective.

Proof. The inverse is defined on products of generators by

$$f \mapsto [s_1 \dots s_t \mapsto \sum_{i=1}^{t} s_1 \dots s_{i-1} f(s_i) s_i \dots s_t].$$

Write $P_1 = A[S]$ for some set S. Consider the diagram



For any $s \in S$, we may write

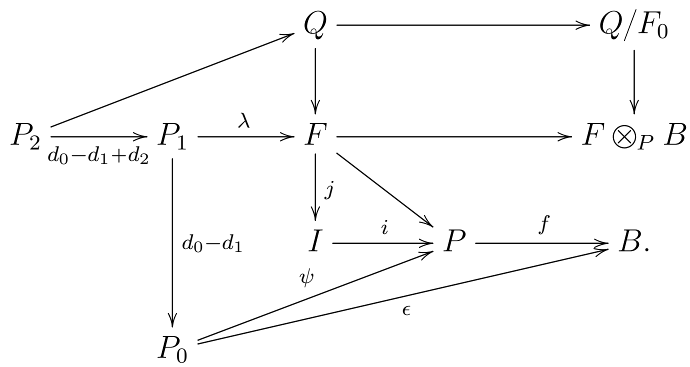
$$\psi(d_0(s) - d_1(s)) = \sum_{t \in T} p_{s,t} f_t \quad \in I$$

for elements $p_{s,t} \in P$ infinitely many of which are zero; choose such $p_{s,t}$ for every $s \in S$ which gives a function $S \to F, s \mapsto (p_{s,t})_{t \in T}$. But the maps

$$(\psi \circ d_0, \psi \circ d_1) : P_1 \rightrightarrows P_0 \to P \circlearrowright F$$

define a (P_1, P_1)-bimodule structure on $F = \oplus_{t\in T} P$, $P = A[S]$, hence by Lemma 4.3 there is a unique biderivation $\lambda : P_1 \to F$ such that $\lambda(s) = (p_{s,t})_t$. We obtain the following diagram:

(2)



Note that $\psi \circ (d_0 - d_1) = i \circ j \circ \lambda$ by Lemma 4.3, because both maps are biderivations and they agree on $S \subset P_1$. □

Step 2: Map in degree 0

Our map of A-modules $\psi : P_0 \to P$ induces a map $d\psi : \Omega_{P_0} \otimes_{P_0} P \to \Omega_{P/A}$ of P-modules hence a map

$$d\psi \otimes 1 : \Omega_{P_0/A} \otimes_{P_0} B \to \Omega_{P/A} \otimes_P B$$

of B-modules.

Step 3: Map in degree 1

From Diagram (2) we see that there is a map $P_1 \to F \otimes_P B$ which is a priori an A-biderivation, but since the (P_1, P_1)-bimodule structure over A on $F \otimes_P B$ is induced by its B-module structure and the maps $P_1 \rightrightarrows P_0 \to B$ which agree because $P_1 \xrightarrow{d_0-d_1} P_0 \to B$ is exact, it follows that the two P_1-module structures on B over $F \otimes_P B$ agree, and therefore the (P_1, P_1)-bimodule structure on $F \otimes_P B$ over A is just a P_1-module structure. This implies that the A-biderivation $P_1 \to F \to F \otimes_P B$ is a usual A-derivation, corresponding to a morphism of P_1-modules $\Omega_{P_1/A} \to F \otimes_P B$ inducing a morphism of B-modules

$$\Omega_{P_1/A} \otimes_{P_1} B \to F \otimes_P B.$$

Step 3: Map in degree 2

Diagram (2) shows that $\lambda(d_0 - d_1 + d_2)(P_2) \subset Q$ because

$$j \circ \lambda \circ (d_0 - d_1 + d_2) = \psi \circ (d_0 - d_1) \circ (d_0 - d_1 + d_2) = \psi \circ 0 = 0.$$

On the other hand, we have seen that Q/F_0 is a B-module, hence a P_2-module via an arrow $P_2 \to B$ defined by one of the arrows in the composite

$$P_2 \Rrightarrow P_1 \rightrightarrows P_0 \xrightarrow{\epsilon} B.$$

Indeed, one can calculate, using the relations between the $d_i \circ d_j$, that no matter what composite you choose above, the arrow $P_2 \to B$ is the same. Now consider the map

$$P_2 \xrightarrow{\lambda\circ(d_0-d_1+d_2)} Q \to Q/F_0.$$

For $f, g \in P_2$, we have

$$\lambda(d_0 - d_1 + d_2)(fg) = \lambda d_0(f)d_0(g) - \lambda d_1(f)d_1(g) + \lambda d_2(f)d_2(g)$$
$$= d_0(f)\cdot\lambda(d_0(g))+\lambda(d_0(f))\cdot d_0(g)-d_1(f)\cdot\lambda(d_1(g))-\lambda(d_1(f))\cdot d_1(g)+d_2(f)\cdot\lambda(d_2(g))+\lambda(d_2(f))\cdot d_2(g)$$
$$= f\cdot\lambda(d_0(g)) + g\cdot\lambda(d_0(f)) - f\cdot\lambda(d_1(g)) - g\cdot\lambda(d_1(f)) + f\cdot\lambda(d_2(g)) + g\cdot\lambda(d_2(f))$$
$$= f(\lambda(d_0(g) - d_1(g) + d_2(g))) + g(\lambda(d_0(f) - d_1(f) + d_2(f))) \qquad \mathrm{mod}\ F_0.$$

In other words, our A-linear map $P_2 \to Q/F_0$ is an A-derivation for the P_2-module structure on Q/F_0. This implies that we obtain a P_2 linear map $\Omega_{P_2/A} \to Q/F_0$ hence a B-linear map

$$\Omega_{P_2/A} \otimes_{P_2} B \to Q/F_0.$$

Step 4: Morphism of complexes

The result is the following diagram:

$$\begin{array}{ccccccc} \Omega_{P_3/A}\otimes_{P_3} B & \longrightarrow & \Omega_{P_2/A}\otimes_{P_2} B & \longrightarrow & \Omega_{P_1/A}\otimes_{P_1} B & \longrightarrow & \Omega_{P_0/A}\otimes_{P_0} B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q/F_0 & \longrightarrow & F\otimes_P B & \longrightarrow & \Omega_{P/A}\otimes_P B. \end{array} \tag{3}$$

We leave it to the reader to verify it commutes. The fact that this induces $H_0(\mathbb{L}_{B/A}) \xrightarrow{\sim} H_0(L)$ and $H_1(\mathbb{L}_{B/A}) \xrightarrow{\sim} H_1(L)$ is not difficult and follows from the identification of $\tau_{\geqslant -1}\mathbb{L}_{B/A}$ with the naive cotangent complex $\mathrm{NL}_{B/A}$ (Theorem 2.1.(6)) which is $(I/I^2 \to \Omega_{P/A}\otimes_P B)$ [Sta18, Tag 00S1]. The isomorphism $H_2(\mathbb{L}_{B/A}) \xrightarrow{\sim} H_2(L)$ is [And74, p. 206, Proposition 12]. □

5. The T^i functors

Next we write $\mathbb{L}_\bullet = \mathbb{L}_{B/A} = \Omega_{P_\bullet/A}\otimes_{P_\bullet} B$ and consider it as a cochain complex of B-modules.

Definition 5.1. We define a functor

$$T^i(B/A, -) : \mathrm{Mod}_B \to \mathrm{Comp}(\mathrm{Mod}_B) \to \mathrm{Mod}_B$$
$$M \mapsto \mathrm{Hom}_B(\mathbb{L}_\bullet, M) \mapsto H^i(\mathrm{Hom}_B(\mathbb{L}_\bullet, M)) =: T^i(B/A, M).$$

Proposition 5.2. *Let $A \to B$ be a ring map with kernel $I \subset A$, and let M be a B-module.*

1. *We have $T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M)$. In particular, $T^0(B/A, B) = T_{B/A}$, the tangent module of B/A.*
2. *We have $T^1(B/A, M) = \mathrm{Exal}_A(B, M)$, the isomorphism classes of extensions of B by M as A-algebras.*
3. *If $A \to B$ is surjective, then $T^0(B/A, M) = 0$ and $T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$. In particular, $T^1(B/A, B) = Hom_B(I/I^2, B) = N_{B/A}$, the normal bundle of B/A.*
4. *If A is noetherian, $A \to B$ of finite type, and M a finite B-module, then $H^i(\mathbb{L}_{B/A})$ and $T^i(B/A, M)$ are finite B-modules.*

Proof. (1) We have $\mathrm{Der}_A(B, M) = \mathrm{Hom}_B(\mathbb{L}_{B/A}, M)$ by Theorem 2.1.(3). But then $\mathrm{Hom}_B(\mathbb{L}_{B/A}, M) =$

$$\{f : \mathbb{L}_0 \to M : (\mathbb{L}_1 \xrightarrow{d_0 - d_1} \mathbb{L}_0 \xrightarrow{f} M) = 0\} = \mathrm{Ker}(\mathrm{Hom}_B(\mathbb{L}_0, M) \to \mathrm{Hom}_B(\mathbb{L}_1, M)) = T^0(B/A, M).$$

(2) We have $\operatorname{Exal}_A(B, M) = \operatorname{Ext}^1_B(\mathbb{L}_{B/A}, M)$ by Theorem 2.1.(4). Hence

$$\operatorname{Exal}_A(B, M) = \operatorname{Ext}^1_B(\mathbb{L}_{B/A}, M) = \operatorname{Hom}_{D(B)}(\mathbb{L}_{B/A}, M[1]) = T^1(B/A, M).$$

(3) In this case we use Theorem 4.1 and take the Schlessinger-Lichtenbaum exact sequence $L_\bullet$, for which we can take $P = A$ so that $L_0 = \Omega_{P/A} \otimes_P B = 0$, hence $T^0(B/A, M) = 0$. Moreover, tensoring the exact sequence of $P = A$-modules

$$0 \to Q \to F \to I \to 0$$

with B gives a diagram for which the horizontal row is exact:

$$\begin{array}{ccccccc} Q \otimes_A B & \longrightarrow & L_1 = F \otimes_A B & \longrightarrow & I/I^2 & \longrightarrow & 0. \\ \downarrow & \nearrow & & & & & \\ L_2 = Q/F_0 & & & & & & \end{array}$$

This shows that

$$0 \to \operatorname{Hom}_B(I/I^2, M) \to \operatorname{Hom}_B(L_1, M) \to \operatorname{Hom}_B(L_2, M)$$

is exact.

(4) See [Sta18, Tag 08PZ] and [Har10, Remark 3.10.1].

□

Theorem 5.3. *Let $A \to B$ be a morphism of rings. Then $T^i(B/A, -) : \operatorname{Mod}_B \to \operatorname{Mod}_B$ is an additive functor, and if*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of B-modules, then there is a long exact sequence of B-modules

$$\begin{gathered} 0 \to \operatorname{Der}_A(B, M') \to \operatorname{Der}_A(B, M) \to \operatorname{Der}_A(B, M'') \to \\ \to \operatorname{Exal}_A(B, M') \to \operatorname{Exal}_A(B, M) \to \operatorname{Exal}_A(B, M'') \to \\ \to T^2(B/A, M') \to T^2(B/A, M) \to T^2(B/A, M'') \to \dots \end{gathered}$$

Proof. By construction the $T^i(B/A, -)$ are additive, and given a short exact sequence as above, since all the $\Omega_{P_n/A} \otimes_{P_n} B$ are free B-modules (Remark 2.2), we get an exact sequence of complexes

$$0 \to \operatorname{Hom}_B(\mathbb{L}_\bullet, M') \to \operatorname{Hom}_B(\mathbb{L}_\bullet, M) \to \operatorname{Hom}_B(\mathbb{L}_\bullet, M'') \to 0.$$

Taking cohomology yields the desired long exact sequence, using the snake lemma for the connecting morphism, and Proposition 5.2 for the identification of $T^0(B/A, -)$ with $\operatorname{Der}_A(B, -)$ and $T^1(B/A, -)$ with $\operatorname{Exal}_A(B, -)$. □

Theorem 5.4. *Let $A \to B \to C$ be ring homomorphisms, and let M be a C-module. Then there is an exact sequence of C-modules*

$$\begin{gathered} 0 \to \operatorname{Der}_B(C, M) \to \operatorname{Der}_A(C, M) \to \operatorname{Der}_A(B, M) \to \\ \to \operatorname{Exal}_B(C, M) \to \operatorname{Exal}_A(C, M) \to \operatorname{Exal}_A(B, M) \to \\ \to T^2(C/B, M) \to T^2(C/A, M) \to T^2(B/A, M) \to \dots \end{gathered}$$

Proof. This simply follows from Proposition 5.2, the distinguished triangle

$$\mathbb{L}_{B/A} \otimes_B^{\mathbf{L}} C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \to \mathbb{L}_{B/A} \otimes_B^{\mathbf{L}} C[1]$$

in $D_{\geqslant 0}(C)$ and the fact that for any triangulated category $(\mathcal{T}, T)$, abelian category $\mathcal{A}$ and cohomological functor $H : \mathcal{T} \to \mathcal{A}$, if $X \to Y \to Z \to TX$ is a distinguished triangle in $\mathcal{T}$, then

$$\ldots \to H(T(X)) \to H(Z) \to H(Y) \to H(X) \to H(T^{-1}(X)) \to \ldots$$

is exact in $\mathcal{A}$. □

Corollary 5.5. *Let $A = k[x_1, \ldots, x_n]$ and $B = A/I$ for some ideal $I \subset A$. Then for any B-module M there is an exact sequence of B-modules*

$$0 \to \mathrm{Der}_k(B, M) \to \mathrm{Der}_k(A, M) \to Hom_B(I/I^2, M) \to T^1(B/k, M) \to 0$$

and an isomorphism $T^2(B/A, M) \cong T^2(B/k, M)$.

Proof. Apply Theorem 5.4 to the morphisms $k \to A \to B$, use Proposition 5.2 and Theorem 2.1.(1) which implies that $\mathbb{L}_{S/R} = \Omega_{S/R}[0]$ for a smooth ring map $R \to S$ hence $T^i(S/R, -) = 0$ for $i > 0$ for a smooth ring map $R \to S$. □

6. Criteria for smooth and local complete intersection morphisms

Next, consider the cotangent sequence $\mathbb{L}_{X/Y}$ of a morphism of schemes $X \to Y$. One way to construct it is as follows. Let $\mathcal{C}$ be a site and let $\mathcal{A} \to \mathcal{B}$ be a morphism of sheaves of rings. Let $\mathcal{P}_\bullet \to \mathcal{B}$ be the canonical resolution: we have $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$, $\mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]]$, and so on. This defines a functor

$$\mathrm{Alg}_{\mathcal{A}} \to \mathrm{hoSimp}(\mathrm{Alg}_{\mathcal{A}}).$$

We can then define the functor $\mathbb{L}_{-/\mathcal{A}}$ to be the composition

$$\mathrm{Alg}_{\mathcal{A}} \to \mathrm{hoSimp}(\mathrm{Alg}_{\mathcal{A}}) \xrightarrow{\Omega_{-/\mathcal{A}}} \mathrm{hoSimp}(\mathrm{Mod}_{\mathcal{A}}) \to \mathrm{hoCh}_{\leqslant 0}(\mathrm{Mod}_{\mathcal{A}}) \to D_{\geqslant 0}(\mathcal{A}).$$

and observe that, for any $\mathcal{B}$ in $\mathrm{Alg}_{\mathcal{A}}$, $\mathbb{L}_{\mathcal{B}/\mathcal{A}}$ lands in $D_{\geqslant 0}(\mathcal{B})$ because $\Omega_{\mathcal{P}_\bullet/\mathcal{A}}$ is a $\mathcal{P}_\bullet$-module and $\mathcal{P}_\bullet \to \mathcal{A}$ a quasi-isomorphism. Equivalently, $\mathbb{L}_{\mathcal{B}/\mathcal{A}}$ is the complex of $\mathcal{B}$-modules constructed using $\mathbb{L}_{\mathcal{B}/\mathcal{A}}$ associated to the simplicial $\mathcal{B}$-module $\Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet} \mathcal{B}$. If $f : X \to Y$ is a morphism of schemes, its cotangent complex is defined as

$$\mathbb{L}_{Y/X} = \mathbb{L}_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}.$$

We can then define the functors

$$\mathcal{T}^i(Y/X, -) : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$$

as $\mathcal{T}^i(Y/X, \mathcal{F}) = H^i(\mathrm{Hom}_{\mathcal{O}_X}(\mathbb{L}_\bullet, \mathcal{F}))$. For any open affine $V = \mathrm{Spec}\ A \subset Y$ and any open affine $U = \mathrm{Spec}\ B \subset f^{-1}(V) \subset X$ where $\mathcal{F} = \tilde{M}$, we obtain $\mathcal{T}^i(Y/X, \mathcal{F})(U) = T^i(B/A, M)$.

It appears that the vanishing of $\mathcal{T}^1$ characterizes smooth morphisms, whereas the vanishing of $\mathcal{T}^2$ characterizes relative local complete intersection morphisms:

Theorem 6.1. *Let $f : X \to Y$ be a flat morphism of finite type between noetherian schemes.*

(1) *f is smooth if and only if $\mathcal{T}^1(X/Y, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ on X. If this is the case, then $\mathcal{T}^2(Y/X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ on X.*
(2) *f is a relative local complete intersection if and only if $\mathcal{T}^2(Y/X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ on X.*

Proof. See [Har10, Theorem 4.11] and [Har10, Remark 4.13.3]. □

7. Deformations of singularities

Theorem 7.1. *Let k be a field and let B be a k-algebra. Then*

$$\mathrm{Def}_{B/k}(D) \cong T^1(B/k, B),$$

where $\mathrm{Def}_{B/k}(D)$ denotes the isomorphism classes of of deformations of B over the dual numbers D. If B is finite as a k-vector space, then $\mathrm{Def}_{B/k}(D)$ is a finite B-module, hence a finite dimensional k-vector space.

Proof. It follows from Proposition 5.2 that

$$T^1(B/k, B) \cong \mathrm{Exal}_k(B, B).$$

Now suppose that $B' \to B$ is a deformation of B over D. This means that B' is flat over D, so that the exact sequence of D-modules

$$0 \to (\epsilon) \to D \to k \to 0$$

induces an exact sequence of B'-modules $0 \to B' \otimes_D (\epsilon) \to B' \to B' \otimes_D k \cong B \to 0$. But $B' \otimes_D (\epsilon) = B' \otimes_D k \otimes_k (\epsilon) \cong B \otimes_k (\epsilon) \cong B$ as k-modules so that we obtain an exact sequence of k-vector spaces

$$0 \to B \xrightarrow{i} B' \xrightarrow{\pi} B \to 0 \tag{4}$$

where $B \cong B' \otimes_D (\epsilon) \subset B'$ is an ideal of square 0 in B'. Conversely, given an exact sequence (4), we define a D-module structure on B' by $\epsilon \cdot b = i(\pi(b))$. Then $B' \otimes_D (\epsilon) \to B'$, $b \otimes \epsilon \mapsto \epsilon \cdot b$ defines an isomorphism $B' \otimes_D (\epsilon) \cong \epsilon \cdot B' \subset B'$. Hence $(B' \otimes_D k) \otimes_k (\epsilon) = B' \otimes_D (\epsilon) \to B'$ is injective so B' is flat over D by Wouter's lecture (see also [Har10, Proposition 2.2(2)]). Hence $T^1_{B/k} = \mathrm{Exal}_k(B, B)$. The finiteness statement follows from Proposition 5.2.(4). □

Examples 7.2. (1) A cusp $B = k[x, y]/(y^2 - x^3)$ has a 2-dimensional space of deformations $\mathrm{Def}_B(D)$.
(2) An ordinary double point of a surface $B = k[x, y, z]/(xy - z^2)$ has a 1-dimensional space of deformations $\mathrm{Def}_B(D)$.
(3) Let $Y \subseteq \mathbb{P}^5_k$ be the Veronese surface in $\mathbb{P}^5_k$, i.e. the image of the Veronse embedding of $\mathbb{P}^2_k \to \mathbb{P}^5_k$. Then the cone $X \subseteq \mathbb{A}^6_k$ over Y is rigid, i.e. $\mathrm{Def}_X(D) = (0)$.

Proof. (*Sketch*) (1)&(2). Use the identification $\mathrm{Def}_B(D) = T^1_{B/k}$ (Theorem 7.1) and Corollary 5.5. For (3), we need the following:

Lemma 7.3. *Let $Y = \mathrm{Proj}(B) \subset \mathbb{P}^n_k$, $R = k[X_0, \ldots, X_n] \supseteq I, B = R/I$ be a nonsingular projectively normal subvariety of n-dimension projective space over k. If $H^1(\mathcal{O}_Y(d)) = H^1(\mathcal{T}_Y(d)) = 0$ for all $d \in \mathbb{Z}$, then the affine cone $X = \mathrm{Spec}\ B \subset \mathbb{A}^{n+1}_k$ over Y is rigid.*

Proof. Since Y is projectively normal, B is integrally closed hence $\mathrm{depth}_x(B) \geqslant 2$; since $H^1(\mathcal{O}_Y(d)) = 0$ for all d, we actually have $\mathrm{depth}_x(B) \geqslant 3$ [Har10, Remark 5.4.1]. By [Har10, Theorem 5.4], there is an injection $T^1_{B/k} \hookrightarrow \oplus_{d\in\mathbb{Z}} H^1(Y, \mathcal{T}_Y(d))$. But $H^1(Y, \mathcal{T}_Y(d)) = 0$ for all $d \in \mathbb{Z}$, so $T^1_{B/k} = 0$, hence X is rigid by Theorem 7.1. □

We use Lemma 7.3: Y is projectively normal and $H^1(\mathcal{O}_Y(d)) = 0$ for all $d \in \mathbb{Z}$. Use the Euler sequence $0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}^3_{\mathbb{P}^2} \to \mathcal{T}_{\mathbb{P}^2}(-1) \to 0$ to obtain an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_{\mathbb{P}^2}(d+1)^3 \to \mathcal{T}_{\mathbb{P}^2}(d) \to 0$$

hence an exact sequence

$$\ldots \to H^1(\mathcal{O}_{\mathbb{P}^2}(d)) = 0 \to H^1(\mathcal{O}_{\mathbb{P}^2}(d+1)^3) = 0 \to H^1(\mathcal{T}_{\mathbb{P}^2}(d)) \to H^2(\mathcal{O}_{\mathbb{P}^2}(d)) \to H^2(\mathcal{O}_{\mathbb{P}^2}(d+1)^3) \to \ldots$$

from which it follows that $H^1(\mathbb{P}^2_k, \mathcal{T}_{\mathbb{P}^2_k/k}(d)) = H^2(\mathbb{P}^2_k, O_{\mathbb{P}^2}(d)) = k$ if $d = -3$ and zero otherwise, hence $H^1(Y, \mathcal{T}_Y(d)) = H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(2d)) = 0$ for each $d \in \mathbb{Z}$. □

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