RESOLUTION OF SINGULARITIES

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1 Introduction

1.1 **General idea.** The goal of this notes is to explain the main ideas behind the proof of Hironaka's theorem on resolutions of singularities. We work over a field k of characteristic zero.

Theorem 1.1. *Let X be a variety. Then there exists a smooth variety X' and a birational projective morphism of varieties $f\colon X' \to X$.*

1.2 **Rough sketch of the proof.** The steps are as follows:

1. Assume X quasi-projective for simplicity. Embed $X \hookrightarrow P$ for P smooth and $\operatorname{codim}(\bar{X}, P) \geq 2$. Let I be the ideal sheaf of $\bar{X}$. Let $Z \subset P$ smooth such that $\operatorname{maxord}(I) = \operatorname{ord}_Z(I)$. Blow up Z to get $P_1 \to P$ with $I \cdot \mathcal{O}_{P_1} = I_1' \cdot I_1$ where I_1' has vanishing locus contained in a SNC divisor. Prove that it suffices to show that one can repeat this procedure in such a way that $\operatorname{maxord}(I_r) < \operatorname{maxord}(I)$.

   Example 1.2. Blow up $\mathbb{A}^2$ in the origin. Let $I = (x^2)$. Then $\operatorname{ord}_0(I) = 2$. We get $\{(x, y) \mid xz = yw\} \subset \mathbb{A}^2 \times \mathbb{P}^1$. Our ideal transforms into the ideal $y^2 w^2 = 0$.

2. Thus, we need *order reduction*: let X be a smooth variety, and I an ideal. We need to show that there exists a smooth blow up sequence $X_r \to \cdots \to X_0 = X$ such that $I \cdot \mathcal{O}_{X_r} = I_r' \cdot I_r$ with $\operatorname{maxord}(I_r) < \operatorname{maxord}(I)$ (and $V(I_r') \subset E_r$).

3. Key concept: a hypersurface of maximal contact for I. This is a smooth hypersurface $H \subset X$ that, among other things, has the property that for $Z \subset X$ smooth with $Z \subset V(I)$ and $\operatorname{ord}_Z(I) = m$, we have $Z \subset H$, and this remains true after blow-up. They may not exist globally, but they do exist locally.

4. Key idea: Construct an ideal $W(I) \subset \mathcal{O}_X$ such that order reduction for I is equivalent to order reduction of $W(I)$, and such that $W(I)$ behaves very nicely with respect to maximal contact hypersurfaces for $W(I)$. (We may then in fact replace I by $W(I)$ to assume that I itself has nice properties with respect to maximal contact hypersurfaces.) The most important property is roughly speaking: order reduction for $W(I)|_H$ implies order reduction for $W(I)$.

5. Choose, for a cover of opens $U \subset X$, a maximal contact hypersurface H_U for each U with respect to $W(I)|_U$. By induction, we have functorial order reduction for $W(I)|_{H_U}$, and hence functorial order reduction for $W(I)|_U$ by construction.

6. Now $W(I)$ is constructed in such a way that for any two maximal contact hypersurfaces $H, H' \subset X$ for $W(I)$, the hypersurfaces H and H' are étale locally isomorphic in a way that preserves $W(I)$. In particular, étale locally there is an automorphism of X that pulls back H' to H, preserves $W(I)$, and hence gives an isomorphism $\phi\colon H \xrightarrow{\sim} H'$ compatible with $W(I)|_{H'}$ and $W(I)|_H$. The order reduction $H_r \to H$ for $W(I)|_H$ (obtained by induction) is functorial, and hence one obtains an isomorphism of order reductions $(H_r \to H) \cong (H'_r \to H')$. Since the order reductions for $W(I)|_H$ are in equivalence with the order reductions for $W(I)$, the resulting order reductions for $W(I)$ are also equivalent!

7. In particular, we get order reduction for $W(I)|_U$ in a way that does not depend on the chosen maximal contact hypersurface H_U.

8. Let V and U be opens in X, with maximal contact hypersurfaces H_V and H_U for $W(I)|_U$ and $W(I)|_V$. Consider the above constructed functorial order reductions for $W(I)|_U$ and $W(I)|_V$. By functoriality, the order reduction for $W(I)|_{U\cap V}$ is the restriction of the order reduction for $W(I)|_U$, and also of the one for $W(I)|_V$. Thus, they glue!

9. Consequently, these local order reductions glue to an order reduction

$$X_r \to \cdots \to X_0 = X$$

for $W(I)$. By construction of $W(I)$, this is also an order reduction for I. We are done.

2 Different versions of resolution of singularities

Definition 2.1. 1. (Resolution.) Let X be a variety. A *resolution* is a projective birational morphism $f\colon X' \to X$ such that X is smooth.

2. (Strong resolution.) Let X be a variety. A *strong resolution* of X is a projective birational morphism $f\colon X' \to X$ such that
   (i) X is smooth
   (ii) f is an isomorphism over the smooth locus of X
   (iii) $f^{-1}(\mathrm{Sing})(X)$ is a simple normal crossings divisor on X'.

3. (Functorial resolution.) A *functorial resolution* is the datum of a resolution $f_X\colon X' \to X$ for every variety X, such that a smooth morphism $\phi\colon X \to Y$ lifts

to a smooth morphism $\phi'\colon X' \to Y'$ that makes the following diagram cartesian:

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

In particular, this gives a functor from the category of varieties and smooth morphisms to the category of smooth varieties and smooth morphisms.

4. (Embedded resolution.) Let $X \subset Y$ be a closed subvariety of a smooth variety Y. An *embedded resolution* is a sequence of blow ups

$$Y_n \to Y_{n-1} \to \cdots \to Y_0 = Y$$

with smooth centers $Z_i \subset Y_i$ and strict transforms $X_i \subset Y_i$, such that Z_i does not contain any irreducible component of X_i and such that X_n is smooth.

5. ((Functorial) principalization.)

For (X, I), there is a blow up sequence $X_r \to \cdots \to X_1 \to X_0 = X$ in smooth centers $Z_i \subset X_i$, which have simple normal crossings with E_i (defined inductively as the union of the exceptional divisor of $X_i \to X_{i-1}$ with the inverse image of E_{i-1}), such that $\Pi\colon X_r \to X$ is an isomorphism outside $V(I)$, such that $I \cdot \mathcal{O}_{X_r}$ has vanishing locus contained in a simple normal crossings divisor; and such that this commutes with smooth morphisms.

Lemma 2.2. *Functorial principalization implies:*
(i) Strong embedded resolution for quasi-projective varieties.
(ii) Strong functorial resolution for varieties.

Proof. (i) Choose an embedding $X \hookrightarrow \bar{X} \hookrightarrow P$ with P smooth and $\operatorname{codim}(\bar{X}, P) \geq 2$. Let I be the ideal sheaf. Show that a sequence of smooth blow-ups $\Pi\colon P_r \to \cdots \to P$ that makes I_r locally principal, gives a resolution of $\bar{X}$. Namely, "by accident", for some j, $Z_j \subset P_j$ maps onto $\bar{X}$ and $Z_j \to \bar{X}$ is birational. Since Z_j is smooth, this gives a resolution.

(Idea: since $\operatorname{codim}(X, P) \geq 2$, we have that I is not locally principal at η_X. But $\Pi^*(I)$ is locally principal, so η_X lies in the locus of P where $\Pi\colon P' \to P$ is not an isomorphism. Thus, we can lift $\eta_X \in P$ to $\eta_X \in P_j$ for a unique j such that $P_j \to P$ is a local isomorphism at $\eta_X \in P_j$ and $P_{j+1} \to P_j$ is a blow up with center $Z_j \subset P_j$. This is the j that we are looking for.)

(ii) Cover X with open affine subset U, and for eacah U, choose a closed embeddings $U \hookrightarrow \mathbb{A}^n =: Y$. As in the above proof, we get an embedded resolution $Y' = Y_n \to \cdots \to Y_1 \to Y_0 = Y$ of $U \subset Y$. The first thing to show is that the resulting resolution $U' \to U$ does not depend on the embedding of U in any smooth variety; this holds, because the principalization procedure can be assumed to commute with closed embeddings (in an appropriate way), and the fact that any two embeddings $\iota_1\colon U \hookrightarrow \mathbb{A}^n$ and $\iota_2\colon U \hookrightarrow \mathbb{A}^m$ become equivalent under an automorphism of $\mathbb{A}^{n+m}$. This proves the desired resolution for affine varieties, which is functorial because of the functoriality of principalization. It remains to prove that one can glue together a global resolution out of the local resolutions; this follows from functoriality. □

Example 2.3. Consider the cuspidal curve $X \subset Y = \mathbb{A}^2$. We need to do four blow ups to arrive at principalization, whereas only one blow up already sufficed for resolution.

3 From order reduction to principalization

1. (Principalization):

   Let X be a smooth variety and I an ideal sheaf. There exists a functorial smooth blow-up sequence $\Pi\colon X' \to X$ such that $I' = I \cdot \mathcal{O}_{X'}$ becomes monomial.

2. Let $a = \text{maxord}(I)$. Let $Z \subset X$ be smooth such that $\text{ord}_Z(I) = a$. If $X_1' \to X$ is the blow up of Z, then we have the crucial formula

   $$I \cdot \mathcal{O}_{X_1'} = \mathcal{O}(-E_1)^a \cdot I_1 \qquad \text{for some ideal } I_1 \text{ with } \text{maxord}(I_1) \leq a.$$

   In particular, applying successive blow ups in smooth centers $Z_i \subset X_i$ with $\text{ord}_{Z_i}(I_i) = a$ as long as $\text{maxord}(I_i) = a$, we get

   $$I \cdot \mathcal{O}_{X_i} = \mathcal{O}(-E_i)^a \cdot I_i, \qquad \text{maxord}(I_i) \leq a.$$

   If we can reduce $\text{maxord}(I_r)$ (via successive blow-ups) such that $\text{maxord}(I_r) = 0$, then $I\mathcal{O}_{X_r} = \mathcal{O}(-E_r)^a$ is an invertible ideal cosupported on the exceptional divisor $E_r \subset X_r$ of the sequence, defined recursively by $E_{i+1} = \pi_i^{-1}(E_i) + F_i$ where F_i is the exceptional divisor of $\pi_i\colon X_{i+1} \to X_1$. Since E_r is supported on a SNC divisor, we have $I \cdot \mathcal{O}_{X_r} = \mathcal{O}(-E_r)^a$ is therefore monomial, and we win.

4 Proof of order reduction

1. By the above, we are reduced to:

   (Functorial order reduction for ideals.) Let X be a smooth variety, $E_0 \subset X$ a simple normal crossings divisor, and I an ideal sheaf, with $\text{maxord}(I) = m$. Then, there exists a smooth blow up sequence $X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$ with centers $Z_i \subset X_i$ such that $\text{ord}_{Z_i}(I_i) = m$, such that Z_i has normal crossings with E_i for each $i < r$, and such that $\text{maxord}(I_r) < m$.

2. For simplicity, we shall prove:

   (Order reduction for ideals, forgetting about divisors.) Let X be a smooth variety and I an ideal sheaf, with $\text{maxord}(I) = m$. Then, there exists a smooth blow up sequence $X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$ with centers $Z_i \subset X_i$ such that $\text{ord}_{Z_i}(I_i) = m$, such that $\text{maxord}(I_r) < m$.

3. To prove this, we will also need to prove:

   (Order reduction for marked ideals.) Let (I, m) be a marked ideal, such that $\operatorname{maxord}(I) \geq m$. There exists a smooth blow up sequence $X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$ with centers $Z_i \subset X_i$ such that $\operatorname{ord}_{Z_i}(I_i) \geq m$, such that $\operatorname{maxord}(I_r) < m$.

4. One proves that order reduction for ideals in dimension n implies order reduction for marked ideals in dimension n. This is relatively easy.

5. We want to prove order reduction by induction. Let $a = \operatorname{maxord}(I)$. Let $p \in X$ with $\operatorname{ord}_p(I) = a$. We show that, for some open neighbourhood $p \in U \subset X$, there exists a smooth hypersurface $H_U \subset U$ such that for every blow-up sequence $U_r \to U_{r-1} \to \cdots \to U_1 \to U$ in smooth centers $Z_i \subset U_i$ with $\operatorname{ord}_{Z_i}((I_U)_i) = a$, we have $Z_i \subset (H_U)_i$ (=the strict transform of $(H_U)_{i-1}$ under $\pi_i \colon X_i \to X_{i-1}$); we may also assume that this property remains true after restricting to an open subset of U. Such a hypersurface $H_U \subset U$ is called a hypersurface of maximal contact. It seems quite a miracle that these hypersurfaces exist locally.

6. From our ideal $I \subset \mathcal{O}_X$ with $\operatorname{maxord}(I) = m$ we construct an ideal $W(I)$ with $\operatorname{maxord}(W(I)) = m!$ that has the following properties:

   (a) Blow-up sequences of order m for (X, I) are blow-up sequences of order $m!$ for $(X, W(I))$, and conversely. Moreover, for any such a sequence, the maximal order of I drops if and only if the maximal order of $W(I)$ drops.

   (b) For any hypersurface of maximal contact H, blow-up sequences of order $m!$ for $W(I)$ are blow-up sequences of order $\geq m!$ for $(H, W(I)|_H, m!)$ and conversely.

   (c) For any two hypersurfaces of maximal contact $H, H' \subset X$, there are étale surjections $\psi, \psi' \colon U \rightrightarrows X$, such that $\psi^{-1}(H) = (\psi')^{-1}(H')$ and $\psi^*(W(I)) = (\psi')^*(W(I))$.

7. We can now start the proof. By induction, we have order induction for ideals in dimension $n-1$. By Step 4, this implies order induction for marked ideals in dimension $n-1$. Consider the ideal $W(I)$ constructed in Step 6. Pick $p \in X$ with $\operatorname{ord}_p(I) = m = \operatorname{maxord}(I)$. Let $p \in U \subset X$ be an open neighbourhood and $H_U \subset U$ a hypersurface of maximal contact with respect to $W(I)|_U$. By order reduction for marked ideals in dimension $n-1$, we get a functorial blow up sequence

$$(H_U)_r \to \cdots \to (H_U)_1 \to H_U$$

   that reduces the order of $W(I)|_{H_U}$. By construction, this is a functorial order reduction for $W(I)|_U$, which is independent of H by property (6c). By functoriality, these order reductions glue to an order reduction $X_r \to \cdots \to X_0 = X$, and we are done. □