

Prismatic cohomology

Learning seminar

Organized by [Olivier de Gaay Fortman](#)

Nine morning meetings (10:00–12:00), between 22 October–18 December 2026

On Fridays, except for Thursday 22 October and Thursday 3 December

Format: 90 minutes for the talk, including questions, followed by 30 minutes of discussion

Introduction

Prismatic cohomology gives a common source for several cohomology theories; its central achievement is a system of *integral comparison theorems*: one cohomology theory recovers de Rham cohomology of the total space, crystalline cohomology of the special fiber, and p -adic étale cohomology of the geometric generic fiber. Working integrally means that we keep track of p -power torsion, rather than passing to $\mathbf{Q}_p$.

Let $K/\mathbf{Q}_p$ be a finite field extension with ring of integers $\mathcal{O}_K$, residue field k , and uniformizer π . Consider a smooth projective morphism

$$X \longrightarrow S = \mathrm{Spec}(\mathcal{O}_K).$$

The generic fibre $X_\eta = X_K$ is defined over a field of characteristic zero, while the special fibre $X_0 = X_k$ is defined over a field of characteristic p . There are several natural ways to study this family:

Cohomology	Main features
$H_{\mathrm{dR}}^i(X/\mathcal{O}_K)$	Differential forms on the whole family. After inverting p this becomes the de Rham cohomology of the generic fiber X_K , equipped its Hodge filtration.
$H_{\mathrm{crys}}^i(X_0/W(k))$	Crystalline cohomology of the special fiber, equipped Frobenius. Here $W(k)$ is the unramified extension of $\mathbf{Z}_p$ with residue field k .
$H_{\mathrm{ét}}^i(X_{\bar{K}}, \mathbf{Z}_p)$	Étale cohomology of the geometric generic fibre, equipped with the $\mathrm{Gal}(\bar{K}/K)$ action.

Comparing these cohomology groups is a central problem of p -adic Hodge theory. Prismatic cohomology yields a very clean way to do exactly that, while also keeping track of mod- p information (which is invisible in the more classical setting of p -adic Hodge theory, where one usually inverts p in the coefficient rings). The prismatic site puts comparison theories into one framework and yields new integral consequences, thereby constituting a reorganisation of the foundations of integral p -adic Hodge theory.

Goal of this learning seminar

The aim of this learning seminar is to:

1. Learn the basics of the foundations of prismatic cohomology.
2. Study and understand the various comparison theorems.
3. Learn to work with prismatic cohomology by working out some beautiful examples and applications.

A little bit more detail

Consider the smooth projective family $X/\mathcal{O}_K$. One may ask: *Can the above different cohomology groups, together with their additional structures, be recovered from a single integral object?* This leads one to consider Breuil–Kisin prisms. Let $A = W(k)[[u]]$, let $\theta : A \rightarrow \mathcal{O}_K$ send u to π , and write $I = \ker \theta = (E(u))$, with E Eisenstein. Equip A with Witt-vector Frobenius on $W(k)$ and $\varphi_A(u) = u^p$. The pair (A, I) is the *Breuil–Kisin prism*. For the p -adic completion $\widehat{X}$, there are finitely generated A -modules

$$H_{\Delta}^i(\widehat{X}/A).$$

They have cup products and a semilinear Frobenius F , so $F(ax) = \varphi_A(a)F(x)$. The induced map

$$H_{\Delta}^i(\widehat{X}/A) \otimes_{A, \varphi_A} A[1/E] \xrightarrow{\sim} H_{\Delta}^i(\widehat{X}/A)[1/E]$$

is an isomorphism. The different classical theories are recovered by changing coefficients in this cohomology theory. Write $H_{\Delta}^i(\widehat{X}/A; B, f)$ for prismatic cohomology with coefficients changed along $f : A \rightarrow B$: change coefficients in a projective cochain complex computing the theory, then take its i -th cohomology.

1. De Rham comparison. With $\alpha = \theta \circ \varphi_A$, there are natural isomorphisms

$$H_{\Delta}^i(\widehat{X}/A; \mathcal{O}_K, \alpha) \cong H_{\mathrm{dR}}^i(X/\mathcal{O}_K), \quad H_{\mathrm{dR}}^i(X/\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \cong H_{\mathrm{dR}}^i(X_K/K).$$

Thus this specialization recovers differential-form cohomology of the family and its generic fibre.

2. Crystalline comparison. Let $\beta : A \rightarrow W(k)$ send u to 0 and act by Witt-vector Frobenius on $W(k)$. Then

$$H_{\Delta}^i(\widehat{X}/A; W(k), \beta) \cong H_{\mathrm{crys}}^i(X_0/W(k)),$$

compatibly with Frobenius. Changing coefficients further to k , at the cochain level, recovers $H_{\mathrm{dR}}^i(X_0/k)$.

3. Hodge–Tate comparison. The untwisted specialization along θ is governed by the groups of differential forms. Put $L = I/I^2$, an invertible $\mathcal{O}_K$ -module. There is a spectral sequence of cohomology groups

$$E_2^{a,b} = H^a(X, \Omega_{X/\mathcal{O}_K}^b) \otimes_{\mathcal{O}_K} L^{\otimes(-b)} \implies H_{\Delta}^{a+b}(\widehat{X}/A; \mathcal{O}_K, \theta).$$

The terms with $a+b=i$ contribute to degree i . Integral degeneration or a direct-sum decomposition is not being asserted.

4. Étale comparison. Set $C = \widehat{K}$, let $\mathcal{O}_C^b$ be its tilt, and put $C^b = \mathrm{Frac}(\mathcal{O}_C^b)$. The rings $A_{\mathrm{inf}} = W(\mathcal{O}_C^b)$ and $B = W(C^b)$ will be explained in meeting 2. Base change of the family to $\mathcal{O}_C$ gives

$$P_C^i := H_{\Delta}^i(\widehat{X}_{\mathcal{O}_C}/A_{\mathrm{inf}}; B) \cong H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} B.$$

Here the coefficient map is the natural inclusion $A_{\mathrm{inf}} \rightarrow B$. The comparison respects Galois actions; Frobenius on the right acts only on B . Since $B^{F=1} = \mathbf{Z}_p$, it also gives

$$H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbf{Z}_p) \cong (P_C^i)^{F=1},$$

where $F=1$ now means ordinary fixed elements of a cohomology group. Compatible roots of π give the prism map $A \rightarrow A_{\mathrm{inf}}$ linking this comparison to the original theory.

Some references: [BS], Theorem 1.8 and Example 1.9; [BMS2], Theorem 1.2; [BMS1], Theorem 1.8 for the étale comparison; [K], Theorem 23.2.5 for the torsion inequality. See also [FKW], §2.1.

Program and dates

All meetings take place from 10:00 to 12:00. The first meeting and the meeting in the first week of December are on Thursdays; all other meetings are on Fridays.

Date	Topic
1 Thu 22 October	A geometric motivation; the comparison problem
2 Fri 30 October	δ -rings, prisms and the A_{inf} example
3 Fri 6 November	The prismatic site and its cohomology
4 Fri 13 November	Crystalline, Hodge–Tate and de Rham comparisons
5 Fri 20 November	Étale comparison and integral consequences
6 Fri 27 November	Essential dimension I: restriction of mod- p cohomology
7 Thu 3 December	Essential dimension II: incompressibility of covers
8 Fri 11 December	Wild Brauer classes I: the cohomological input
9 Fri 18 December	Wild Brauer classes II: slopes and arithmetic

Part I: foundations

1. Motivation and the comparison problem

Introduce $X/\mathcal{O}_K$, its two fibres and the three cohomology theories on page 1. Explain the Hodge, Frobenius and Galois structures, and why retaining p -torsion matters. Describe the purpose of crystalline thickenings informally. Preview how differential forms constrain mod- p restriction maps and how this leads to the two applications. *Goal of this talk:* describe degree-one cohomology of an elliptic curve with good reduction, and explain what the common prismatic complex must remember. *Scope:* state the comparison package informally; no crystalline-site construction or proofs. *Reading:* [B], I; [K], §1; [FKW] and [ANP], introductions; [N1]. [BdJ] is background.

2. δ -rings, prisms and the A_{inf} example

Define a p -derivation and its Frobenius lift $\varphi(x) = x^p + p\delta(x)$. Recall Witt vectors and define a bounded prism, explaining the Cartier ideal, completeness and $p \in I + \varphi(I)A$. Work through three examples: the crystalline prism $(W(k), (p))$, the Breuil–Kisin prism, and the perfect prism attached to $\mathcal{O}_C$, where $C = \widehat{K}$. For the last example, explain the tilt $\mathcal{O}_C^\flat$, the ring $A_{\text{inf}} = W(\mathcal{O}_C^\flat)$, and Fontaine’s map with its principal kernel. State the perfect-prism/perfectoid correspondence. *Goal of this talk:* verify $(\mathbf{Z}_p[[u]], (u-p))$, distinguish the maps $u \mapsto p, p^p, 0$, and construct $u \mapsto [\pi^p]$ into A_{inf} using compatible roots. Check compatibility with Frobenius and reduction. *Scope:* devote roughly half the talk to definitions and the first two examples, and half to A_{inf} . Take the Witt-vector construction, general derived completeness and the perfectoid correspondence as inputs. Explain derived reduction through one two-term example; introduce $W(C^\flat)$ only as needed for page 2. Free δ -rings and other period rings are optional reading. *Reading:* [B], II–IV; [BS], §§2–3, definitions and examples; [BMS1], §3. [K], §§2–3, 5–8, and [N3] are companions.

3. The prismatic site and its cohomology

For a smooth formal scheme $Y/(A/I)$, define the objects and coverings of $(Y/A)_\Delta$, and its structure and reduced structure sheaves. Explain how their cohomology gives the two complexes. Describe the origin of Frobenius, functoriality and cup products. State completed base change and proper finiteness, which make the comparisons usable. *Goal of this talk:* compute the relative point $Y = \text{Spf}(A/I)$, obtaining the complex A , its Frobenius and its reduction A/I . *Scope:* sketch the role of prismatic envelopes and Čech computations; take their existence and descent properties as inputs. *Reading:* [B], V; [K], §§11–12; [BS], §4 and Theorem 1.8; [N5].

Part II: comparison theorems

4. Crystalline, Hodge–Tate and de Rham comparisons

State the first three comparisons on page 2, keeping the coefficient maps θ, α, β distinct. Explain the Hodge–Tate cohomology sheaves and their global spectral sequence. Use the Bockstein to illustrate how differential forms enter and how the de Rham differential is recovered. *Goal of this talk:* work through the Hodge–Tate map in degrees 0 and 1 for one coordinate, including the twist I/I^2 . *Scope:* prove this local calculation; state the general comparison and gluing results. Full proofs of all three theorems are beyond this meeting. *Reading:* [B], V–VI; [BS], Theorem 1.8 and §§5–6; [N5–N6]. [BdJ] and [K], §§14–15, are background.

5. Étale comparison and integral consequences

State the finite-coefficient comparison over a perfect prism. Explain derived Frobenius fixed points and their elementary characteristic- p model, the Artin–Schreier sequence. Indicate why perfection and descent occur in the proof, taking the deeper results as inputs. Deduce the mod- p dimension inequality from comparison and semicontinuity of a perfect complex. *Goal of this talk:* explain the group-level formula on page 2 and its Galois action. Distinguish this special setting from the general derived fixed-point formula, where kernels alone need not suffice. *Scope:* prove the dimension inequality using the comparison theorem as an input. *Reading:* [B], IX; [K], §§22–23; [BS], §9; [N2, N4].

Part III: applications

6. Essential dimension I: restriction of mod- p cohomology

Use [FKW], §2.1, to translate a restriction map into a map of mod- p prismatic complexes over $k[[u]]$. Explain the small-degree freeness input and the role of Cartier theory in obtaining a lower bound from differential forms. Keep the unramified good-reduction and degree hypotheses visible. *Goal of this talk:* follow the image module through specialization and localization when the open is dense in the special fibre. Explain which comparison is used at each step. *Scope:* give this central argument; state the freeness and Cartier inputs, and outline the additional passage to arbitrary complex opens. *Reading:* [FKW], §§2.1–2.2, especially the proof of the restriction bound; [N1–N2].

7. Essential dimension II: incompressibility of covers

Define essential dimension and p -essential dimension of a finite cover. Explain how characteristic classes from the covering group and the restriction bound obstruct compression to a smaller-dimensional variety. Show how trace handles auxiliary covers of degree prime to p . *Goal of this talk:* explain why the mod- p homology cover of a complex abelian variety is p -incompressible for sufficiently large p , using the cup products of degree-one classes. *Scope:* prove the deduction from the restriction theorem. Leave logarithmic extensions, Heisenberg groups and Shimura varieties for further study. *Reading:* [FKW], introduction and §2.3, especially Corollary 2.3.12 and its inputs.

8. Wild Brauer classes I: the cohomological input

Introduce the Brauer group, the Kummer sequence and evaluation at local points. Explain how restriction maps and vanishing cycles enter [ANP]. Introduce the Nygaard filtration through its relation to divisibility by powers of p under Frobenius, and state the syntomic/vanishing-cycle comparison needed in the argument. *Goal of this talk:* unpack the Frobenius eigenspace and restriction map in [ANP], Theorem 1.2.2, and identify the role of each comparison theorem. *Scope:* explain the additional tools and their use; take their general construction and comparison proofs as inputs. *Reading:* [ANP], §§2–3; [BMS2], §10; [BS], §§12, 15 as background.

9. Wild Brauer classes II: slopes and arithmetic

Introduce Newton slopes and their relation to Hodge numbers, then explain the total-slope argument of [ANP], §4. Combine it with meeting 8 to outline the construction of a Brauer class with surjective evaluation after finite extension. Deduce the weak-approximation consequence, stating all hypotheses. *Goal of this talk:* trace the implication from a nonzero global 2-form to nonconstant Brauer evaluation; use an abelian surface as a guiding geometric example. *Scope:* focus on the main argument and arithmetic deduction. Explicit products of elliptic curves and prismatic Dieudonné theory are optional extensions. *Reading:* [ANP], §§4–5, and optionally §6; [ALB] for the last direction.

Literature

[B] Bhargav Bhatt, *Prismatic cohomology*, Eilenberg Lectures, Columbia University, 2018. An efficient sequence from δ -rings to comparisons; lectures are available separately. [Lecture notes and course page](#)

[K] Kiran S. Kedlaya, *Notes on prismatic cohomology*; UC San Diego course, spring 2021. A detailed companion with exercises, extra background and recorded lectures. [Online notes](#) · [Course page, PDF and recordings](#)

[BS] Bhargav Bhatt and Peter Scholze, *Prisms and Prismatic Cohomology*. This is the foundational paper. Section references in this Program use arXiv version 4 (2022). [arXiv:1905.08229v4](#)

[BdJ] Bhargav Bhatt and Aise Johan de Jong, *Crystalline cohomology and de Rham cohomology* (2011). A short treatment of the crystalline–de Rham comparison. [arXiv:1110.5001](#)

[BMS1] Bhargav Bhatt, Matthew Morrow and Peter Scholze, *Integral p -adic Hodge theory*. Selected reading: §3 (perfectoid rings) and §4 (Breuil–Kisin–Fargues modules). [arXiv:1602.03148](#)

[BMS2] Bhargav Bhatt, Matthew Morrow and Peter Scholze, *Topological Hochschild homology and integral p -adic Hodge theory*. Advanced reading suggested by Ambrosi: §§7.4–8 and §10; Nygaard complexes, characteristic p , and nearby cycles. [arXiv:1802.03261](#)

[FKW] Benson Farb, Mark Kisin and Jesse Wolfson, *Essential dimension via prismatic cohomology*. Main application: introduction and §§2.1–2.2; essential dimension in §2.3. [Duke Mathematical Journal 173\(15\), 2024](#)

[ANP] Emiliano Ambrosi, Rachel Newton and Margherita Pagano, *Wild Brauer classes via prismatic cohomology*. Second application. Start with the introduction; the prismatic arguments are in §§3–4. [Authors' preprint \(version dated 30 October 2025\)](#)

Further directions

[ALB] Johannes Anschütz and Arthur-César Le Bras, *Prismatic Dieudonné theory*. An application to p -divisible groups; suitable for a subsequent series. [arXiv:1907.10525](#)

[BL] Bhargav Bhatt and Jacob Lurie, *Absolute prismatic cohomology* (2022). A later route through the Cartier–Witt stack and prismatic crystals. [arXiv:2201.06120](#)

[E] Matthew Emerton, *Prismatic Cohomology*, Chicago course, spring 2020. Supplementary notes expanding technical points in Bhatt's lectures. [Course page and notes](#)