Hodge classes on abelian varieties of split Weil type

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Introduction

These are talk notes concerned with the role of abelian varieties of split Weil type in the study of Hodge classes on abelian varieties. Although Hodge classes on abelian varieties can be difficult to understand in general, those on abelian varieties of CM type admit a useful description: by a theorem of André, they are pull-backs of special Hodge classes, namely split Weil classes, on abelian varieties of split Weil type.

We begin by recalling the basic notions of CM fields, CM types, and abelian varieties of CM type. We then introduce abelian varieties of split Weil type relative to a CM field E. Roughly speaking, these are abelian varieties A that carry an action of E and whose polarization is induced by a split E-hermitian form. The split condition has two parts: the hermitian form has balanced signature at every complex embedding of E, and its discriminant is $(-1)^n$. Under these hypotheses, the exterior power

$$\bigwedge_E^{2n} V \subset \bigwedge_{\mathbb{Q}}^{2n} V, \qquad V = H^1(A, \mathbb{Q}),$$

consists of Hodge classes. These are the split Weil classes.

We will explain how André's theorem enters Deligne's theorem that Hodge classes on complex abelian varieties are absolute Hodge. The argument has four ingredients. First, one extends a given Hodge class in a family to a section of the local system that is Hodge everywhere on the Hodge locus of that Hodge class. Second, by the density of CM points in the relevant Shimura variety and by Deligne's Principle B, one reduces to the case of abelian varieties of CM type. Third, André's theorem reduces the problem further to split Weil classes. Finally, applying Deligne's Principle B again, this time to a suitable moduli space of abelian varieties of split Weil type, it suffices to consider such abelian varieties that are of the form A_0^d, where $d = [E\colon \mathbb{Q}]$, where the split Weil classes are generated over E by $\{0\} \times A_0^{d-1}$. There, the absoluteness is immediate.

We also recall some consequences of these ideas: by work of Milne, the Hodge conjecture for complex abelian varieties of CM type implies the standard conjectures for abelian varieties in arbitrary characteristic as well as the Tate conjecture for abelian varieties over finite fields.

1 CM abelian varieties and abelian varieties of split Weil type

1.1 **CM abelian varieties.** We start with some definitions.

Definition 1.1. A *CM field* is a number field $E/\mathbb{Q}$ equipped with an involution $E \to E, e \mapsto \bar{e}$ such that $\overline{s(e)} = s(\bar{e})$ for each embedding $s\colon E \to \mathbb{C}$. A *CM type* is a function $\varphi\colon \mathrm{Hom}(E, \mathbb{C}) \to \{0, 1\}$ such that $\varphi(s) + \varphi(\bar{s}) = 1$ for each $s \in \mathrm{Hom}(E, \mathbb{C})$.

Definition 1.2.
1. Let A be a simple abelian variety. We say that A is of *CM type* if $E = \mathrm{End}^0(A)$ is a CM field over which $H_1(A, \mathbb{Q})$ has dimension one.
2. An abelian variety A is of CM type if A is isogenous to a product of simple abelian varieties of CM type.

Definition 1.3. Let $S = \mathrm{Hom}(E, \mathbb{C})$. We denote by E_φ the rational Hodge structure of type $(1, 0)$, $(0, 1)$ whose underlying $\mathbb{Q}$-vector space is E and whose Hodge decomposition is defined as follows:

$$E_\varphi \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{s\in S} (E \otimes_{\mathbb{Q}} \mathbb{C})_s, \qquad (E_\varphi \otimes_{\mathbb{Q}} \mathbb{C})^{1,0} = \bigoplus_{s\in\varphi^{-1}(1)} \mathbb{C}, \qquad (E_\varphi \otimes_{\mathbb{Q}} \mathbb{C})^{0,1} = \bigoplus_{s\in\varphi^{-1}(0)} \mathbb{C}.$$

After the appropriate Tate twist, this gives a Hodge structure of type $\{(-1, 0), (0, -1)\}$; equivalently, it is the homology Hodge structure of the CM abelian variety attached to the CM type φ. Concretely, one can define $\Phi = \varphi^{-1}(1)$ and consider the complex torus $\mathbb{C}^\Phi/\Phi(\mathcal{O}_E)$, where Φ now also denotes the map $E \to \mathbb{C}^\Phi$ defined as $\Phi(e) = (\varphi(e))_{\varphi\in\Phi}$.

Corollary 1.4. *Let A be a complex abelian variety. Then A is of CM type if and only if $H^1(A, \mathbb{Q}) \simeq E_{1,\varphi_1} \oplus \cdots \oplus E_{r,\varphi_r}$ for some CM types $(E_i, \varphi_i\colon \mathrm{Hom}(E_i, \mathbb{C}) \to \{0, 1\})$.*

Remark 1.5. Let A be a simple abelian variety. Then $\mathrm{End}^0(A)$ is a field over which $H_1(A, \mathbb{Q})$ has dimension one if and only if $\mathrm{MT}(A)$ is commutative.

Proof. Let us assume that A is simple and that $E = \mathrm{End}^0(A)$ is a number field of degree $d = [E\colon \mathbb{Q}]$ such that $V = H_1(A, \mathbb{Q})$ is a one-dimensional E-vector space. If $\mathscr{C}(X)$ is the commutant of a simple subalgebra $X \subset \mathrm{End}_{\mathbb{Q}}(V)$, then, as $\mathrm{End}_{\mathbb{Q}}(V)$ is a finite-dimensional central simple algebra over $\mathbb{Q}$, we have by the double centralizer theorem that $\mathscr{C}(\mathscr{C}(X)) = X$ and $\dim_{\mathbb{Q}}(X) \cdot \dim_{\mathbb{Q}}(\mathscr{C}(X)) = \dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}}(V))$.

We apply this to $E \subset \mathrm{End}_{\mathbb{Q}}(V)$, for which we have $E \subset \mathscr{C}(E)$ as E is commutative. Since $\dim_{\mathbb{Q}}(V) = \dim_E(V) \cdot [E\colon \mathbb{Q}] = [E\colon \mathbb{Q}] =: d$, we get

$$d \cdot \dim_{\mathbb{Q}}(\mathscr{C}(E)) = \dim_{\mathbb{Q}}(E) \cdot \dim_{\mathbb{Q}}(\mathscr{C}(E)) = \dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}}(V)) = (\dim_{\mathbb{Q}}(V))^2 = d^2.$$

It follows that $\dim_{\mathbb{Q}}(\mathscr{C}(E)) = d$ and since $E \subset \mathscr{C}(E)$ we get $E = \mathscr{C}(E)$. We have an embedding of algebraic groups $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E}) \hookrightarrow \mathrm{GL}(V)$, on $\mathbb{Q}$-points given by the action of E^* on V. Now $\mathrm{End}^0(A) \subset \mathrm{End}_{\mathbb{Q}}(V)$ consists of those endomorphisms of V that commute with $\mathrm{MT}(A)$. Thus, $\mathrm{MT}(A)$ is contained in the commutant of $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E})$ which, by the above, equals $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E})$. Therefore, $\mathrm{MT}(A)$ is commutative.

Conversely, assume $\mathrm{MT}(A)$ is commutative. Let $V = H_1(A, \mathbb{Q})$ and $E = \mathrm{End}^0(A)$. The algebra E is a division algebra by the general theory of abelian varieties. For a non-zero element $v \in V := H_1(A, \mathbb{Q})$, we have a map $E \to V$ sending $e \in E$ to $e \cdot v \in V$; this map is injective because if $e \in E$ would be a a non-zero element in the kernel then $e \cdot v = 0$ which implies that $v = (e^{-1}e) \cdot v = e^{-1} \cdot (e \cdot v) = 0$, a contradiction. Hence $\dim_{\mathbb{Q}} V \geq \dim_{\mathbb{Q}} E$. We then consider the decomposition $V_{\mathbb{C}} =$

$\oplus V_\chi$ of $V_\mathbb{C}$ into character spaces (so χ ranges over the characters $\mathrm{MT}(A) \to \mathbb{G}_m$ and $V_\chi = \{v \in V_\mathbb{C} \mid a \cdot v = \chi(a)v \;\forall\, a \in \mathrm{MT}(A)\}$). Let $g \in \mathrm{MT}(A)(\mathbb{C})$ and $h \in \mathrm{End}(V_\chi)$ for some χ. We have $gh = hg$ because both preserve each piece in the character decomposition, and $gh(v_\chi) = \chi(g)h(v_\chi) = h(\chi(g)v_\chi)$ for $v_\chi \in V_\chi$. Therefore, $\mathrm{End}(V_\chi)$ commutes with $\mathrm{MT}(A)$ and is therefore contained in $E = \mathrm{End}^0(A) \subset \mathrm{End}_\mathbb{Q}(V)$. This gives $\oplus_\chi \mathrm{End}(V_\chi) \subset E \otimes_\mathbb{Q} \mathbb{C} \subset \mathrm{End}_\mathbb{C}(V \otimes \mathbb{C})$, hence

$$\dim_\mathbb{Q}(E) \geq \sum_\chi (\dim_\mathbb{C} V_\chi)^2 \geq \sum_\chi (\dim_\mathbb{C} V_\chi) = \dim_\mathbb{Q} V \geq \dim_\mathbb{Q} E. \tag{1}$$

Thus, $\dim_\mathbb{Q} V = \dim_\mathbb{Q} E$. It also follows that $\dim_\mathbb{C} V_\chi = 1$ and

$$E \otimes_\mathbb{Q} \mathbb{C} = \bigoplus_\chi \mathrm{End}(V_\chi) = \bigoplus_\chi \mathbb{C}.$$

Therefore, E is commutative, and hence a field (since it is a division algebra). As $\dim_\mathbb{Q}(V) = \dim_\mathbb{Q}(E)$ by (1), we have $\dim_E(V) = 1$ and are done. $\square$

Remark 1.6. Assume that A is a simple CM abelian variety. As the above proof shows, we have that $\mathrm{MT}(A)$ is a torus. One can show that $E = \mathrm{End}^0(A)$ is a CM field and the Rosati involution defined by any polarization on A is complex conjugation.

1.2 **Abelian varieties of split Weil type.** We will now study abelian varieties of split Weil type. For this, we need the notion of split hermitian form over a CM field.

Definition 1.7. Let V be an E-vector space. An E-valued form $\phi\colon V \times V \to E$ is said to be *E-hermitian* if $\phi(e{\cdot}v, w) = e{\cdot}\phi(v, w)$ and $\overline{\phi(v, w)} = \phi(w, v)$ for all $v, w \in V, e \in E$. For each embedding $s\colon E \to \mathbb{C}$, we get a hermitian form $\phi_s\colon (V_\mathbb{C})_s \times (V_\mathbb{C})_s \to \mathbb{C}$. Let (a_s, b_s) be its signature. We get a hermitian form on the one-dimensional E-vector space $\wedge_E^d V$ which must be of the form $(x, y) \mapsto f x \bar{y}$ when we choose a basis vector, and we define

$$\mathrm{disc}(\phi) := f \in F^*/\mathrm{Nm}_{E/F}(E^*).$$

We say that ϕ is *split* if ϕ is non-degenerate, if $d = 2n = \dim_E(V)$ is even, $a_s = b_s = d/2 = n$ for each $s \in S$, and $\mathrm{disc}(\phi) = (-1)^n$.

Example 1.8. Define $\phi\colon E^d \times E^d \to E$ as

$$\phi(x, y) = x_1\bar{y}_1 + \cdots + x_{d/2}\bar{y}_{d/2} - x_{d/2+1}\bar{y}_{d/2+1} + \cdots + x_d\bar{y}_d.$$

One can show that this is essentially the only example (every split hermitian form is equivalent to this standard one). Indeed, this follows from the following remark.

Remark 1.9.

1. A theorem of Landherr implies that the data $(\{(a_s, b_s)\}, \mathrm{disc}(\phi))$ determine a non-degenerate E-hermitian form ϕ up to isomorphism.
2. Let ϕ be non-degenerate. Then ϕ is split if and only if there exists a totally isotropic subspace of V of dimension $d/2$.

Example 1.10. $E = \mathbb{Q}(\sqrt{-d})$ and consider the hermitian matrix $\phi\colon E^2 \times E^2 \to E$ defined by the matrix

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the diagonal embedding $E \hookrightarrow E^2$ gives a totally isotropic subspace of dimension one. The matrix

$$\phi' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has the same invariants, hence Landherr's theorem implies that ϕ and ϕ' are equivalent. And indeed, we can conjugate the first matrix by $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ to obtain the second matrix.

Let E be a CM field and let $S = \mathrm{Hom}(E, \mathbb{C})$. Let V be a rational Hodge structure of type $\{(1,0),(0,1)\}$ such that

$$E \subset \mathrm{End}_{\mathrm{Hdg}}(V).$$

Let $2n = \dim_E(V)$. The action of E decomposes the complexification of V as follows. Corresponding to the decomposition $E \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{s\in S} \mathbb{C}$ we have a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{s\in S} (V_{\mathbb{C}})_s,$$

where $e \in E$ acts on $(V_{\mathbb{C}})_s$ as multiplication by $s(e)$.

Definition 1.11. We say that V is *of split Weil type relative to E* if $\dim_E(V)$ is even and its polarization is induced by a split E-hermitian form.

We now explain what this means. Since E respects the Hodge decomposition, we get induced decompositions

$$(V_{\mathbb{C}})_s = (V_{\mathbb{C}})_s^{1,0} \oplus (V_{\mathbb{C}})_s^{0,1} \qquad \text{with} \qquad \dim_{\mathbb{C}}(V_{\mathbb{C}})_s^{1,0} + \dim_{\mathbb{C}}(V_{\mathbb{C}})_s^{0,1} = \dim_{\mathbb{C}}(V_{\mathbb{C}})_s = d.$$

Lemma 1.12. *There exists a canonical bijection $\phi \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(\phi)$ between:*

1. *The set of alternating forms $E\colon V \times V \to \mathbb{Q}$ such that $E(a\cdot x, y) = E(x, \bar{a}\cdot y)$.*
2. *The set of sesquilinear forms $\phi'\colon V \times V \to E$.*

Proof. This follows from the non-degeneracy of the trace pairing $E \times E \to \mathbb{Q}$. □

Lemma 1.13. *Let $\eta \in E^*$ be a totally imaginary element. Then $\phi \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(\eta\phi)$ defines a canonical bijection between:*

1. *The set of non-degenerate E-hermitian forms $\phi\colon V \times V \to E$.*
2. *The set of non-degenerate alternating forms $\psi\colon V \times V \to \mathbb{Q}$.*

If $\phi \leftrightarrow \psi$ under this bijection, then we have

$$s(\eta)\psi_{\mathbb{C}}(x, \bar{y}) = \phi_s(x, y) \qquad \forall x, y \in (V_{\mathbb{C}})_s \subset V \otimes_{\mathbb{Q}} \mathbb{C}.$$

In particular, if $s(\eta) > 0$, then $i\psi_{\mathbb{C}}(x, \bar{y})$ is positive definite on $V_s^{1,0}$ if and only if $\phi_s(x, y)$ is positive definite on $V_s^{1,0}$ if and only if $\psi_s(x, y)$ is negative definite on $(V_{\mathbb{C}})_s^{0,1}$.

Proof. The main point is that $s(\eta)\psi_{\mathbb{C}}(x,\bar{y}) = \phi_s(x,y)$, which we leave to the reader. □

Corollary 1.14. *Let $\eta \in E$ such that $\eta^2 \in F$ and $\bar{\eta} = -\eta$. Then $\phi \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(\eta\phi)$ defines a bijection between:*

1. *The set of non-degenerate E-hermitian forms $\phi\colon V \times V \to E$ such that ϕ_s is positive definite on $(V_{\mathbb{C}})_s^{1,0}$ for each $s \in S = \mathrm{Hom}(E,\mathbb{C})$ such that $\Im(s(\eta)) > 0$.*

2. *Alternating forms $\psi\colon V \times V \to \mathbb{Q}$ such that $i\psi_{\mathbb{C}}(x,\bar{y})$ is positive definite on $V^{1,0}$.*

In particular, the signature $\{(a_s,b_s)\}_{s\in S}$ of the hermitian form ϕ is given as follows:

$$a_s = \dim_{\mathbb{C}}(V_{\mathbb{C}})_s^{1,0} \qquad \text{and} \qquad b_s = \dim_{\mathbb{C}}(V_{\mathbb{C}})_s^{0,1} \qquad (s \in S \mid \Im(s(\eta)) > 0)\,. \tag{2}$$

Proof. Consider an alternating form $\psi\colon V \times V \to \mathbb{Q}$ such that $i\psi_{\mathbb{C}}(x,\bar{y})$ is positive definite on $V^{1,0}$, and let $\phi\colon V \times V \to E$ be the corresponding E-hermitian form. As $\phi_s(x,y) = s(\eta)\psi_{\mathbb{C}}(x,\bar{y})$, we get that for $x \in (V_{\mathbb{C}})_s^{0,1}$,

$$\phi_s(x,x) = \overline{\phi_s(x,x)} = \phi_{\bar{s}}(\bar{x},\bar{x}) = \bar{s}(\eta)\psi_{\mathbb{C}}(\bar{x},x) < 0.$$

To see the latter inequality, note that $\bar{x} \in V_{\bar{s}}^{1,0}$, that $i\psi_{\mathbb{C}}(x,\bar{y})$ is positive definite there, and that $\bar{s}(\eta) < 0$. We conclude that ϕ_s is positive definite on $(V_{\mathbb{C}})_s^{1,0}$ and negative definite on $(V_{\mathbb{C}})_s^{0,1}$. Therefore, as (a_s,b_s) is the signature of ϕ_s, we get:

$$\dim(V_{\mathbb{C}})_s^{1,0} \leq a_s \qquad \text{and} \qquad \dim(V_{\mathbb{C}})_s^{0,1} \leq b_s.$$

Since $\dim(V_{\mathbb{C}})_s^{1,0} + \dim(V_{\mathbb{C}})_s^{0,1} = d = \dim_E(V) = a_s + b_s$, the equalities (2) follow. □

Remark 1.15. Let $E\colon V \times V \to \mathbb{Q}$ be a polarization such that $E(ax,y) = E(x,\bar{a}y)$. Let $\eta \in E^*$ such that $\bar{\eta} = -\eta$ and let ϕ be the hermitian form attached to E. Note that the condition that $(a_s,b_s) = (d/2,d/2)$ does not depend on the polarization, because $(a_s,b_s) = (\dim(V_{\mathbb{C}})_s^{1,0}, \dim(V_{\mathbb{C}})_s^{0,1})$. However, we also require that $\mathrm{disc}(\phi) = (-1)^{d/2}$.

Proposition 1.16. *Let V be of split Weil type relative to E. Then the subspace*

$$\bigwedge_E^{2n} V \subset \bigwedge_{\mathbb{Q}}^{2n} V$$

consists entirely of Hodge classes. They are called split Weil classes*.*

Proof. Indeed,

$$\left(\bigwedge_E^{d} V\right) \otimes_{\mathbb{Q}} \mathbb{C} = \bigwedge_{E\otimes_{\mathbb{Q}}\mathbb{C}}^{d} V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{s\in S} \bigwedge_{\mathbb{C}}^{d} (V_{\mathbb{C}})_s = \bigoplus_{s\in S} \bigoplus_{p_s+q_s=d} \left(\bigwedge_{\mathbb{C}}^{p_s} (V_{\mathbb{C}})_s^{1,0}\right) \otimes \left(\bigwedge_{\mathbb{C}}^{q_s} (V_{\mathbb{C}})_s^{0,1}\right)$$
$$= \bigoplus_{s\in S} \left(\bigwedge_{\mathbb{C}}^{d/2} (V_{\mathbb{C}})_s^{1,0}\right) \otimes \left(\bigwedge_{\mathbb{C}}^{d/2} (V_{\mathbb{C}})_s^{0,1}\right).$$

This shows that every such class has Hodge type (n,n), as claimed. □

Remark 1.17. We have $\dim_E(V) = 2n$ hence $\dim_{\mathbb{Q}}(V) = 2n \cdot [E : \mathbb{Q}]$. If A is an abelian variety of split Weil type relative to E, with $H_1(A, \mathbb{Q}) = V$, then these classes appear in $H^{2n}(A, \mathbb{Q})$. This is not the middle degree when $e = [E : \mathbb{Q}] > 2$. Indeed, with $e = [E : \mathbb{Q}]$ one has

$$g = \dim(A) = n \cdot [E : \mathbb{Q}] = en, \qquad 2g = 2ne,$$

whereas the split Weil classes lie in degree $2n$.

The following converse, due to André, explains why the condition of being of split Weil type is exactly what is detected by the existence of a non-zero Hodge class in the top exterior power over E, in the case of an abelian variety of split Weil type which is isogenous to a product of simple abelian varieties of CM type with respect to E.

Lemma 1.18 (André)**.** *Let $E_{\varphi_1}, \ldots, E_{\varphi_{2n}}$ be CM types on a CM field E. Define $V = \bigoplus E_{\varphi_i}$ (which is an E-vector space of dimension $2n$ and a Hodge structure of CM type). Assume there exists a non-zero element*

$$0 \neq x \in \bigwedge_E^{2n} V \subset \bigwedge_{\mathbb{Q}}^{2n} V$$

such that x is a Hodge class. Then V is of split Weil type relative to E.

Proof. We first prove the signature condition, namely that $\dim(V_s^{1,0}) = \dim(V_s^{0,1}) = n$ for each $s \in S$. We have $\bigwedge_E^{2n} V \simeq E$ and the Hodge structure is given by

$$\left(\bigwedge_E^{2n} V\right) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{s \in S} \mathbb{C},$$
$$\left(\left(\bigwedge_E^{2n} V\right) \otimes_{\mathbb{Q}} \mathbb{C}\right)^{p,q} = \bigoplus_{\sum_{i=1}^{2n} \varphi_i(s) = p} \mathbb{C}.$$

The element $x \in \bigwedge_E^{2n} V$ defines an element $x \otimes 1 \in \left(\bigwedge_E^{2n} V\right) \otimes \mathbb{C}$ which yields in the above decomposition an element

$$(x_s)_s \in \bigoplus_{s \in S} \mathbb{C}.$$

Since x is non-zero as an element of the one-dimensional E-vector space $\bigwedge_E^{2n} V$, each component x_s is non-zero. On the other hand, x is a Hodge class, and therefore

$$\bigoplus_{s \in S} \mathbb{C} = \bigoplus_{s \in S | \sum_i \varphi_i(s) = n} \mathbb{C}.$$

That is, we have $\sum_{i=1}^{2n} \varphi_i(s) = n$ for each $s \in S$.

Next, observe that

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{i=1}^{2n} E_{\varphi_i} \otimes \mathbb{C} = \bigoplus_{i=1}^{2n} \bigoplus_{s \in S} \mathbb{C},$$

with

$$(V \otimes_{\mathbb{Q}} \mathbb{C})^{1,0} = \bigoplus_{i=1}^{2n} \bigoplus_{s \in S\colon \varphi_i(s)=1} \mathbb{C}.$$

Hence $V_s^{1,0} = \bigoplus_{i|\varphi_i(s)=1} \mathbb{C}$ and therefore,

$$\dim(V_s^{1,0}) = \sum_{i|\varphi_i(s)=1} \varphi_i(s) = \sum_{i=1}^{2n} \varphi_i(s) = n.$$

Exactly the same computation shows that $\dim(V_s^{0,1}) = n$.

It remains to define a polarization Q on V such that $Q(ax, y) = Q(x, \bar{a}y)$ and such that, for a suitable $\eta \in E$ with $\bar{\eta} = -\eta$, $Q = \mathrm{Tr}_{E/\mathbb{Q}}(\eta\phi)$ for some hermitian $\phi\colon V \times V \to E$ with $\mathrm{disc}(\phi) = (-1)^n$.

To do this let, for each i, $\eta_i \in E$ be such that $\Im(s(\eta_i)) > 0$ for each $s \in S$ such that $\varphi_i(s) = 1$. This yields a polarization $\psi_i(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\eta_i x\bar{y})$ on E_{φ_i} and hence a polarization

$$\psi = \sum_i \psi_i$$

on $V = \oplus_i E_{\varphi_i}$. Let $\eta = \eta_{2n}$ and define

$$\phi(x, y) = \sum_{i=1}^{2n} \eta^{-1}\eta_i x_i \bar{y}_i.$$

Then $\phi\colon V \times V \to E$ is a hermitian form and

$$\psi(x, y) = \sum_{i=1}^{2n} \psi_i(x_i, y_i) = \sum_{i=1}^{2n} \mathrm{Tr}_{E/\mathbb{Q}}(\eta_i x\bar{y}) = \mathrm{Tr}_{E/\mathbb{Q}}\left(\eta \sum_{i=1}^{2n} \eta^{-1}\eta_i x_i \bar{y}_i\right) = \mathrm{Tr}_{E/\mathbb{Q}}(\eta\phi(x, y)).$$

Note that

$$\mathrm{sgn}(s(\mathrm{disc})) = (-1)^{b_s} = (-1)^n$$

for each $s \in S$ hence

$$\mathrm{disc}(\phi) = (-1)^n \cdot f$$

for some totally positive $f \in F^*$. Defining $\phi'(x, y) = \sum_{i=1}^{2n-1} \eta^{-1}\eta_i x_i \bar{y}_i + f^{-1}\eta^{-1}\eta_{2n} x_{2n} \bar{y}_{2n}$, we get

$$\mathrm{disc}(\phi') = f^{-1}\mathrm{disc}(\phi) = (-1)^n$$

and ϕ'_s is still positive definite on $V_s^{1,0}$ for each $s \in S$ such that $\Im(s(\eta)) > 0$, since that was the case for ϕ_s and f is totally positive. In particular, $\psi'(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\eta\phi'(x, y))$ defines a polarization ψ' on V with the required properties, and we are done. □

2 Theorems

Theorem 2.1 (André)**.** *Let A be a complex abelian variety of CM type and let $k \geq 1$ be an integer. Then there exists a CM field E, abelian varieties of split Weil type*

$A_1, \ldots, A_r$ relative to E, such that for each i we have $\dim_E H_1(A_i, \mathbb{Q}) = 2n$ (that is, $\dim(A_i) = n \cdot e$), and morphisms $f_i \colon A \to A_i$ such that the morphism

$$\bigoplus_{i=1}^{r} \mathrm{HdgWeil}^{2n}(A_i, \mathbb{Q}) \to \mathrm{Hdg}^{2n}(A, \mathbb{Q})$$

is surjective.

It is useful to record the following equivalent formulation in the language of rational Hodge structures.

Theorem 2.2. *Let V be a rational Hodge structure of type $\{(1,0),(0,1)\}$ which is of CM type and let $n \geq 1$ be an integer. Then there exists a CM field E, finitely many rational Hodge structures V_α of split Weil type relative to E, with $\dim_E V_\alpha = 2n$, and morphisms of Hodge structure $V_\alpha \to V$ such that the map*

$$\bigoplus_{\alpha} \mathrm{HdgWeil}(\wedge^{2n}_{\mathbb{Q}}(V_\alpha)) \to \mathrm{Hdg}\left(\wedge^{2n}_{\mathbb{Q}} V\right)$$

is surjective.

Corollary 2.3. *The algebraicity of split Weil classes on abelian varieties of split Weil type implies the Hodge conjecture for abelian varieties of CM type.*

As we will show, Theorem 2.1 can also be used to prove the following result.

Theorem 2.4 (Deligne)**.** *Hodge classes on abelian varieties are absolute Hodge.*

Recall the four standard conjectures in the form relevant here:

1. Lefschetz type Standard Conjecture (Conj B): the Lefschetz operator is induced by an algebraic cycle. This is true for abelian varieties by Lieberman, Kleiman.

2. Künneth type Standard Conjecture (Conj C): the Künneth projectors are algebraic. True for abelian varieties by Deninger–Murre.

3. Conjecture D: numerical equivalence vs. homological equivalence. (True for complex abelian varieties by Lieberman.)

4. The Hodge Standard Conjecture. If this holds, then the Lefschetz conjecture implies Conjecture D.

Theorem 2.5 (Milne)**.** *Assume the Hodge conjecture holds for complex abelian varieties of CM type. Then both Conjecture D and the Hodge Standard Conjecture hold for abelian varieties in arbitrary characteristic.*

Remark 2.6. In particular, all the standard conjectures would be true for all abelian varieties over all algebraically closed fields. Indeed, the Lefschetz operator is algebraic by Lieberman, Kleiman, and the Künneth projectors are algebraic by Deninger–Murre. (Alternatively, for the latter, consider the following trick that Thomas Agugliaro explained to me. Let Γ be the graph of $2 \colon A \to A$. Note that 2 acts as 2^j on $H^j(A)$. Fix i and define $Z' \coloneqq \prod_{j \neq i} (\Gamma - 2^j \Delta)$, where the product is the composition of correspondences. Then Z' acts as 0 on $H^j(A)$ for $j \neq i$, and as $\prod_{j\neq i}(2^i - 2^j)$ on $H^i(A)$. Thus, $Z \coloneqq \prod_{j \neq i}(2^i - 2^j)^{-1} Z'$ acts as 0 on $H^j(A)$ for $j \neq i$ and as the identity on $H^i(A)$.)

Theorem 2.7 (Milne)**.** *The Hodge conjecture for complex abelian varieties of CM type implies the Tate conjecture for all abelian varieties over finite fields.*

3 André's theorem

Definition 3.1. For any Hodge structure V_0 of type $(1,0),(0,1)$, define another Hodge structure $V = V_0 \otimes_{\mathbb{Q}} E$ of type $(1,0),(0,1)$ via the canonical isomorphisms

$$V \otimes_{\mathbb{Q}} \mathbb{C} = V_0 \otimes_{\mathbb{Q}} (E \otimes_{\mathbb{Q}} \mathbb{C}) = \bigoplus_{s \in S} V_0 \otimes_{\mathbb{Q}} \mathbb{C}.$$

Proof of Theorem 2.2. We have $V \simeq E_{\varphi_1} \oplus \cdots \oplus E_{\varphi_n}$ for some CM fields E_i with CM types φ_i. Let E be the Galois closure of the compositum of the E_i. Up to replacing V by $V \otimes_{\mathbb{Q}} E$, in which V is a direct summand, we may assume that $E_i = E$ for each i.

We have

$$V \otimes_{\mathbb{Q}} E = \left(\bigoplus_{i=1}^{r} E_{\varphi_i} \right) \otimes_{\mathbb{Q}} E = \bigoplus_{i=1}^{r} \bigoplus_{g \in G} E_{g\varphi_i}.$$

For $\alpha \subset I \times G$ a subset of cardinality $2n$, define $V_\alpha = \bigoplus_{(i,g)\in\alpha} E_{g\varphi_i}$. Then we have a composition

$$V_\alpha \hookrightarrow \bigoplus_{(i,g)} E_{g\varphi_i} = V \otimes_{\mathbb{Q}} E = \bigoplus_{g \in G} V \to V,$$

where $\oplus_{g\in G} V \to V$ is the projection onto the factor corresponding to $e \in G$. This yields maps

$$V_\alpha \to V, \qquad \alpha \subset I \times G \colon |\alpha| = 2n.$$

Moreover,

$$\left(\bigwedge_{\mathbb{Q}}^{2n} V \right) \otimes_{\mathbb{Q}} E = \bigwedge_{E}^{2n} V \otimes_{\mathbb{Q}} E = \bigwedge_{E}^{2n} \left(\bigoplus_{i,g} E_{g\varphi_i} \right) = \bigoplus_{\alpha} \bigotimes_{(i,g)\in\alpha} E_{g\varphi_i} = \bigoplus_{\alpha} \bigwedge_{E}^{2n} V_\alpha$$

Then

$$\bigwedge_{\mathbb{Q}}^{2n} V \quad \text{is a direct summand of} \quad \bigoplus_{g\in G} \bigwedge_{\mathbb{Q}}^{2n} V = \left(\bigwedge_{\mathbb{Q}}^{2n} V \right) \otimes_{\mathbb{Q}} E = \bigoplus_{\alpha} \bigwedge_{E}^{2n} V_\alpha$$

hence every Hodge class $x \in \wedge_{\mathbb{Q}}^{2n} V$ is a sum of images of Hodge classes $x_\alpha \in \bigwedge_E^{2n} V_\alpha$ under the maps

$$\bigwedge_{E}^{2n} V_\alpha \to \left(\bigwedge_{\mathbb{Q}}^{2n} V \right) \otimes_{\mathbb{Q}} E \to \bigwedge_{\mathbb{Q}}^{2n} V$$

which are also the maps $\wedge_E^{2n} V_\alpha \to \wedge_{\mathbb{Q}}^{2n} V$ induced by the maps $V_\alpha \to V$ defined above.

The naive approach now would be to show that all the Hodge structures V_α of dimension $2n$ over E are of split Weil type. However, we will show a finer statement, which is enough for our purposes: as $x \in \wedge_{\mathbb{Q}}^{2n} V$ is a sum of images of Hodge classes $x_\alpha \in \bigwedge_E^{2n} V_\alpha$, it only suffices to show that V_α is of split Weil type whenever $\wedge_E^{2n} V_\alpha$ contains a non-zero Hodge class. This is exactly the content of Lemma 1.18. $\square$

4 Deligne's theorem

We now turn to Deligne's theorem that Hodge classes on abelian varieties are absolute. Let X be a smooth projective complex variety. For an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$, define $X^\sigma = X \times_{\mathbb{C},\sigma} \mathbb{C}$. Recall the canonical isomorphisms

$$\begin{aligned} H^\bullet(X,\mathbb{C}) &\simeq H^\bullet_{dR}(X,\mathbb{C}), \quad \text{which is linear, and} \\ H^\bullet_{dR}(X,\mathbb{C}) &\simeq H^\bullet_{dR}(X^\sigma,\mathbb{C}), \quad \text{which is } \sigma\text{-linear.} \end{aligned}$$

Together, they give a σ-linear isomorphism

$$H^\bullet(X,\mathbb{C}) \simeq H^\bullet(X^\sigma,\mathbb{C}) \quad (\sigma \in \mathrm{Aut}(\mathbb{C})) \quad \text{that we denote by} \quad x \mapsto x^\sigma. \tag{3}$$

Definition 4.1. A class $\alpha \in H^{2n}(X,\mathbb{C})$ is *absolute Hodge* if $\alpha^\sigma \in H^{2n}(X^\sigma,\mathbb{C})$ is a rational Hodge class for each $\sigma \in \mathrm{Aut}(\mathbb{C})$. Since (3) is compatible with the Hodge filtrations, if α itself is a Hodge class, then α is an absolute Hodge class if and only if $\alpha^\sigma \in H^{2n}(X^\sigma,\mathbb{C})$ lies in $H^{2n}(X^\sigma,\mathbb{Q}) \subset H^{2n}(X^\sigma,\mathbb{C})$ for each $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Remarks 4.2. 1. The Hodge conjecture implies Hodge classes are absolute Hodge.

2. In fact, the Hodge conjecture also implies that each integral Hodge class modulo torsion remains integral (modulo torsion) after conjugating by any $\sigma \in \mathrm{Aut}(\mathbb{C})$. The reason is that $H^\bullet(X,\mathbb{Q}) \cap \cap_\ell H^\bullet_{\text{et}}(X,\mathbb{Z}_\ell)/\text{tors} = H^\bullet(X,\mathbb{Z})/\text{tors}$.

Example 4.3. Let V_0 be a rational Hodge structure of even rank $2n$ and of type $(1,0),(0,1)$. Then the Hodge structure $V = V_0 \otimes_\mathbb{Q} E$ is of split Weil type relative to E. Moreover, all the classes in the subspace

$$\bigwedge_E^d V \subset \bigwedge_\mathbb{Q}^d V,$$

which are Hodge by Proposition 1.16, are in fact absolute Hodge. Indeed, $V_0 = H^1(A_0,\mathbb{Q})$ for an abelian variety A_0 of dimension n, so that

$$\bigwedge_E^{2n}(V_0 \otimes_\mathbb{Q} E) = \left(\bigwedge_\mathbb{Q}^{2n} V_0\right) \otimes_\mathbb{Q} E = H^{2n}(A_0,\mathbb{Q}) \otimes_\mathbb{Q} E$$

and $H^{2n}(A_0,\mathbb{Q})$ is generated by the fundamental class of a point. A choice of a basis $E \simeq \mathbb{Q}^e$ defines an isogeny $A \sim A_0^e$ with $e = [E:\mathbb{Q}]$ and the map $H^{2n}(A_0,\mathbb{Q}) \otimes E \simeq H^{2n}(A_0,\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}^e = H^{2n}(A_0,\mathbb{Q})^e \to H^{2n}(A,\mathbb{Q})$ is the sum of the pull-backs along the various projections $A \sim A_0^e \to A_0$.

Theorem 4.4 ("Principle B", Deligne)**.** *Let S be a smooth projective connected variety. Let $\pi\colon X \to S$ be a smooth projective morphism. Fix $s \in S(\mathbb{C})$. Let $\alpha \in H^0(S, R^{2n}\pi_*\mathbb{Q})$ such that $\alpha_s \in H^{2n}(X_s,\mathbb{Q})$ is an absolute Hodge class. Then $\alpha_t \in H^{2n}(X_t,\mathbb{Q})$ is an absolute Hodge class for each $t \in S(\mathbb{C})$.*

Proof. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, consider the conjugated family $\pi^\sigma\colon X^\sigma \to S^\sigma$. View α as a flat section of the relative De Rham cohomology. Assume first that the variant of the statement for Hodge classes is true. We need to show that α_t^σ is Hodge for each $t \in S(\mathbb{C})$ and each $\sigma \in \mathrm{Aut}(\mathbb{C})$. We know that α_s^σ is Hodge for each $\sigma \in \mathrm{Aut}(\mathbb{C})$. We get a flat section α^σ of the conjugated relative De Rham cohomology, and we have $(\alpha^\sigma)_{\sigma(s)} = \alpha_s^\sigma$ is Hodge. Hence $(\alpha^\sigma)_{\sigma(t)} = \alpha_t^\sigma$ is Hodge for each $\sigma(t) \in S^\sigma(\mathbb{C})$.

For the rest of the proof we admit the global invariant cycle theorem; it remains to show that α_t is Hodge for each $t \in S(\mathbb{C})$. For this, choose a smooth compactification $X \hookrightarrow \bar{X}$ and look at the inclusion $i_s\colon X_s \hookrightarrow \bar{X}$, whose pullback yields a morphism of Hodge structures

$$i_s^*\colon H^{2n}(\bar{X}, \mathbb{Q}) \to H^{2n}(X_s, \mathbb{Q})^{\pi_1(S,s)}.$$

This is surjective by the global invariant cycle theorem. Choosing a polarization on the Hodge structure $H^{2n}(\bar{X}, \mathbb{Q})$ we get an isomorphism of Hodge structures

$$\mathrm{Ker}(i_s)^\perp \xrightarrow{\sim} H^{2n}(X_s, \mathbb{Q})^{\pi_1(S,s)}.$$

Since α_s is monodromy invariant, there is a Hodge class $\beta \in \mathrm{Ker}(i_s)^\perp \subset H^{2n}(\bar{X}, \mathbb{Q})$ such that $i_s^*(\beta) = \alpha_s$. As $\alpha_t = i_t^*(\beta)$ we conclude that α_t is a Hodge class. □

Theorem 4.5 (Deligne)**.** *Let $\alpha \in H^{2n}(X, \mathbb{Q})$ be a Hodge class on an abelian variety X. Then α is an absolute Hodge class.*

Step 1 *Claim: There exists a smooth quasi-projective Shimura variety Z and a family $X \to Z$ of polarized abelian varieties such that*

(a) $A = X_t$ for some $t \in Z$

(b) $\alpha \in H^{2n}(X_t, \mathbb{Q})$ extends to a section $\tilde{\alpha} \in H^0(Z, R^{2n}\pi_*\mathbb{Q})$ which is Hodge everywhere on Z.

We construct a family $\mathcal{X} \to Z \subset \mathcal{A}_{g,\delta,N}$ for some g, δ, N by pulling back the universal family over $\mathcal{A}_{g,\delta,N}$ to a suitable subvariety $Z \subset \mathcal{A}_{g,\delta,N}$. We define Z as the Hodge locus defined by the Mumford-Tate group of X. More precisely, there are finitely many Hodge tensors $\tau_1, \cdots, \tau_r$ for $H^1(X, \mathbb{Q})$ such that $\mathrm{MT}(X)$ is exactly the subgroup of $\mathrm{Aut}(V, \psi)$ fixing every τ_i, and we define $Z \subset \mathcal{A}_{g,\delta,N}$ as the irreducible component of the Hodge locus of $\tau_1, \ldots, \tau_r$ that passes through $[X] \in \mathcal{A}_{g,\delta,N}$. By Cattani–Deligne–Kaplan, Z is a quasi-projective variety.

If $D_h \subset \mathbb{H}_g$ denotes the subdomain parametrizing Hodge structures whose Mumford-Tate group is contained in $\mathrm{MT}(X)$. Then $D_h \to Z$ is the pull-back of $\mathbb{H}_g \to \mathcal{A}_{g,\delta,N}$, hence Z is connected and smooth. We have $X = \mathcal{X}_y$ for some $y \in Z(\mathbb{C})$.

Since Z is contained in the Hodge locus of α, up to replacing Z by a finite étale cover $Z' \to Z$, α extends to a section $\tilde{\alpha} \in H^0(Z, R^{2n}f_*\mathbb{Q})$ which is Hodge everywhere on Z. Note that this holds because the connected component of the identity G^0 of the algebraic monodromy group G is contained in the Mumford–Tate group (Deligne), and is normal in M^{der}. Moreover, G/G^0 is finite.

Step 2 To prove that $\alpha = \tilde{\alpha}_t$ is absolute Hodge, it suffices by Principle B to prove that $\tilde{\alpha}_t$ is absolute Hodge for some $t \in Z$. As Z is a Shimura variety, it contains a dense subset of CM points. Hence, we may assume that $A = X_t$ is a complex abelian variety of CM type.

Step 3 We thus assume that X is of CM type and $\alpha \in H^{2n}(X, \mathbb{Q})$ is a Hodge class. We must show that α is absolute Hodge. By André's theorem, we are reduced to prove that each split Weil class on an abelian variety of split Weil type is absolute Hodge. Thus we now assume

$$(X, E \hookrightarrow \mathrm{End}^0(X), \phi)$$

is an abelian variety of split Weil type relative to some CM field, and that $\alpha \in H^{2n}(X, \mathbb{Q})$ is a split Weil class.

Step 4 *Claim: There exists a smooth quasi-projective Shimura variety Z and a family $X \to Z$ of polarized abelian varieties such that*

(a) $A = X_t$ for some $t \in Z$

(b) $\alpha \in H^{2n}(X_t, \mathbb{Q})$ extends to a section $\tilde{\alpha} \in H^0(Z, R^{2n}\pi_*\mathbb{Q})$ which is Hodge everywhere on Z.

(c) For every $t \in Z$, the abelian variety A_t is of split Weil type relative to E.

(d) There exists $t \in Z$ such that $H^1(X_t, \mathbb{Q}) \simeq V_0 \otimes_{\mathbb{Q}} E$ for some Hodge structure V_0 of type $\{(1,0),(0,1)\}$ and even rank.

We proceed as before. We let $\mathcal{X} \to \mathcal{A}_{g,\delta,N}$ be the universal family, where the polarization type is the same as the polarization type of A. We let $D^{sp} \subset D = \mathbb{H}_{ne}$ be the subdomain that classifies Hodge structures of split Weil type. Let $Z \subset \mathcal{A}_{g,\delta,N}$ be the locus of abelian varieties whose endomorphism algebra contains E. Then Z is a Hodge locus, hence a quasi-projective variety; moreover, the preimage in $D = \mathbb{H}_g$ is exactly D_{sp}. Hence Z is smooth. One can show that there exists $t \in Z$ such that $H^1(X_t, \mathbb{Q}) \simeq V_0 \otimes_{\mathbb{Q}} E$ for some Hodge structure V_0 of type $\{(1,0),(0,1)\}$ and even rank. Notice that α remains Hodge everywhere on Z. Indeed, it remains a split Weil class everywhere on Z and therefore Hodge by Proposition 1.16. Up to replacing Z by a finite cover $Z' \to Z$, we can assume that α extends to a section $\tilde{\alpha} \in H^0(Z, R^{2n}\pi_*\mathbb{Q})$. Thus $\alpha = \tilde{\alpha}_t$.

Step 5 By construction, there is a point $s \in Z$ such that the abelian variety $X_s = \pi^{-1}(s)$ satisfies $H^1(A_s, \mathbb{Q}) \simeq V_0 \otimes_{\mathbb{Q}} E$. By Example 4.3, we get that the split Weil class $\tilde{\alpha}_s \in H^{2n}(A_s, \mathbb{Q})$ is an absolute Hodge class. Then Principle B applies and hence $\alpha = \tilde{\alpha}_t$ is absolute Hodge.