COMBINATORICS AND HODGE THEORY OF DEGENERATIONS OF ABELIAN VARIETIES: A SURVEY OF THE MUMFORD CONSTRUCTION

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Abstract. We survey the Mumford construction of degenerating abelian varieties, with a focus on the analytic version of the construction, and its relation to toric geometry. Moreover, we study the geometry and Hodge theory of multivariable degenerations of abelian varieties associated to regular matroids, and extend some fundamental results of Clemens on 1-parameter semistable degenerations to the multivariable setting.

Contents

1. Introduction 2
1.1. Contents 3
1.2. Index of constructions 5
1.3. Acknowledgements 6
2. Preliminary material 6
2.1. Algebraic and analytic stacks 6
2.2. Principally polarized abelian varieties 6
2.3. Degenerations of PPAVs 8
2.4. Toric varieties 14
2.5. Toroidal extensions 16
3. The Mumford construction 20
3.1. Mumford construction, fan version 20
3.2. Weight filtration and dual complex 27
3.3. Mumford construction, polytope version 28
3.4. Comparison of polytope and fan constructions 33
3.5. Examples 34
3.6. Base change and Veronese embedding 40
4. Regular matroids 44
4.1. Matroids, graphs, and quadratic forms 44
4.2. Mumford degenerations associated to regular matroids 47

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4.3. Shifted and transversely shifted matroidal degenerations 49
5. Nodal and semistable morphisms over higher-dimensional bases 53
5.1. Mumford degenerations with nodal singularities 53
5.2. Weight filtration of a semistable morphism 56
5.3. Resolution of the base change of a nodal morphism 60
5.4. Resolution of the base change of a transversely shifted matroidal degeneration 64
6. The second Voronoi fan and Alexeev's theorem 66
6.1. The universal family of abelian torsors with theta divisor 66
6.2. Algebraicity and projectivity 73
7. Proof of Theorem 1.1 77
References 79

1. Introduction

In 1972, Mumford gave an analytic construction of degenerations of abelian varieties over complete rings [44]. It played an important role in the development of the theory of toric varieties [36] and toroidal compactifications of locally symmetric varieties [9]. When working over $\mathbb{C}$, the basic idea is that one may view a degeneration of abelian varieties analytically, as the quotient of an appropriate intermediate cover of the universal cover.

It is well-known that, over the complex numbers, any abelian g-fold $A \simeq \mathbb{C}^g/H_1(A,\mathbb{Z})$ is the quotient of a vector space $\mathbb{C}^g$ by a lattice $H_1(A,\mathbb{Z}) \subset \mathbb{C}^g$ of rank $2g$. Suppose that $A = X_t$ is a general fiber of a degenerating family $X \to \Delta$ of abelian varieties over the unit disk $\Delta \subset \mathbb{C}$. Let $V_\mathbb{Z} := H_1(X_t,\mathbb{Z})$. Then, there is a saturated sublattice $W_{-1}V_\mathbb{Z} \subset V_\mathbb{Z}$ of the fundamental group $V_\mathbb{Z} = \pi_1(X_t)$, consisting of 1-cycles which are invariant under the monodromy of the punctured family $X^* \to \Delta^*$. It contains a further sublattice $W_{-2}V_\mathbb{Z} \subset W_{-1}V_\mathbb{Z}$ consisting of *vanishing cycles*; that is, the 1-cycles on X_t which are null-homologous in X. The filtration $W_{-2}V_\mathbb{Z} \subset W_{-1}V_\mathbb{Z} \subset W_0V_\mathbb{Z} = V_\mathbb{Z}$ defines the *weight filtration*.

The subgroup $W_{-1}V_\mathbb{Z} \subset V_\mathbb{Z} = \pi_1(X_t)$ gives rise to a cover $Y^* \to X^*$ corresponding fiberwise to the intermediate cover $\mathbb{C}^g \simeq \widetilde{X}_t \to Y_t \to X_t$ of the universal cover, whose Galois group over X_t is the graded piece $\mathrm{gr}_0^W V_\mathbb{Z}$. In general, Y_t is a *semiabelian variety*—an algebraic torus bundle over an abelian variety of dimension $\frac{1}{2}\mathrm{rank}\,\mathrm{gr}_{-1}^W V_\mathbb{Z}$. When $W_{-1}V_\mathbb{Z}$ has rank g, then we have $Y_t \simeq \mathbb{C}^g/\mathbb{Z}^g \simeq (\mathbb{C}^*)^g$, and we call the degeneration *maximal*.

In the case of a maximal degeneration, the intermediate cover Y^* of the punctured family admits an analytic open embedding $Y^* \hookrightarrow (\mathbb{C}^*)^g \times \mathbb{C}^*$ into an algebraic torus, with the map to Δ^* given by the projection to the second factor. Thus, by the theory of toric varieties, it is possible to extend $Y^* \to \Delta^*$ to a family $Y \to \Delta$. More precisely, we take a toric extension of $(\mathbb{C}^*)^g \times \mathbb{C}^*$ for which the fiber-preserving action of $\mathrm{gr}_0^W V_\mathbb{Z}$ extends to an action on the central fiber Y_0. The quotient of $Y \to \Delta$ by the extended action of $\mathrm{gr}_0^W V_{\mathbb{Z}}$ produces a new model of the degeneration with particularly nice properties, e.g. toroidal singularities.

The original paper [44] performs this construction more generally over any complete ring, corresponding to a possibly higher-dimensional base, and using formal algebraic geometry. Mumford's work was inspired by an influential 1959 manuscript of Tate [60] on degenerating elliptic curves. In the late 1970's, Nakamura and Namikawa worked out the theory in the complex-analytic setting [47, 48, 49, 50, 51], culminating in a method of patching together Mumford constructions over the cones of the second Voronoi fan, to produce a relatively proper, analytic extension of the universal abelian variety $\mathcal{X}_g \to \mathcal{A}_g$ to a toroidal compactification of $\mathcal{A}_g$.

An impressive, and notoriously technical, further advancement was the work of Faltings–Chai [17] in the early 1990's, who extended the Mumford construction and the theory of toroidal compactifications to the arithmetic setting. In the later 1990's, Alexeev–Nakamura [7] and Alexeev [4] used the Mumford construction to compactify the universal abelian torsor $(\mathcal{X}_g^\star, \Theta_g)$ with theta divisor, cf. Construction 6.6, and provide a modular interpretation of this compactification, as the normalization of the closure of the space of KSBA-stable pairs $(X, \epsilon\Theta)$ in the proper DM stack of log general type varieties.

1.1. **Contents.** The goal of this paper is to provide a "working mathematician's guide" to the Mumford construction. Thus, we usually work analytically over $\mathbb{C}$, though we do also touch on the question of algebraicity of Mumford constructions (Prop. 6.12). Furthermore, we largely restrict our attention to maximal degenerations, though most of the results presented here apply in the non-maximal case. Many of the ideas of the paper are to be found scattered through the literature; some are difficult to find, and others appear to be new, such as Theorem 1.1.

After reviewing in Section 2 preliminary material on principally polarized abelian varieties, their Hodge theory, their moduli, their degenerations, and toroidal extensions and compactifications of $\mathcal{A}_g$, we dive into the main constructions in Section 3:

Mumford constructions and examples. Using tools from toric geometry, we construct maximal degenerations of principally polarized abelian g-folds. The constructions are presented in increasing levels of generality and are broadly divided into two classes: *fan* constructions and *polytope* constructions (see Section 1.2 for a list). The fan construction only produces a degeneration complex-analytically, but has the advantage of being relatively simpler, and having more readable geometry. The advantage of the polytope construction is that it outputs a relatively projective degeneration.

The equality of the two constructions is examined in Section 3.4, while the topology is discussed in Section 3.2, where we analyze the weight filtration on the limiting mixed Hodge structure of a general fiber. We also study the effect, on Mumford constructions, of a toroidal base change and of replacing the polarization by a multiple, see Section 3.6.

Regular matroids. We continue in Section 4 with a more detailed analysis of Mumford constructions associated to regular matroids (Constr. 4.16). A *matroid* $\underline{R}$ is a collection of subsets of a finite set E, encoding the notion of linear independence, of a set of vectors in a vector space (Def. 4.1). An embedding $E \hookrightarrow \mathbb{F}^n$ into a vector space over a field $\mathbb{F}$, realizing this collection of independent sets, is a *realization* of $\underline{R}$. Matroids which admit a realization over any field are *regular* (Def. 4.2), though $\mathbb{F}_2$ and $\mathbb{F}_3$ suffice.

Associated to a regular matroid $\underline{R}$ are the so-called *matroidal*, *shifted matroidal*, and *transversely shifted matroidal* Mumford constructions, see Sections 4.2 and 4.3. Related degenerations were explored for cographic matroids by Dancso–McBreen–Shende [22, Sec. 8.3], building on unpublished work of Hausel–Proudfoot. Perhaps unsurprisingly, regular matroids are intimately connected to the total space of a Mumford construction being regular, i.e. smooth. In fact, a Mumford construction $X \to \Delta^k$ such that (i) X is regular, and (ii) over each coordinate hyperplane $\{u_i = 0\}$, $i = 1, \ldots, k$, of the polydisk, the vanishing cycles span a 1-dimensional space, is necessarily a transversely shifted matroidal degeneration, and vice versa (Props. 5.3 and 5.13).

As we explain in Section 2.3, for a family $f^*\colon X^* \to (\Delta^*)^k$ of g-dimensional PPAVs, the monodromy about the i-th coordinate hyperplane defines, via the principal polarization, a symmetric matrix $B_i \in \mathrm{Sym}_{g\times g}(\mathbb{Z})$ (Def. 2.6). The cone in $\mathrm{Sym}_{g\times g}(\mathbb{R})$ generated by $\{B_1, \ldots, B_k\}$ is the *monodromy cone* of the family f^* (Def. 2.7). Such a cone is *matroidal* if it is induced by an integral realization of a regular matroid (Def. 4.6). Transversely shifted matroidal degenerations are examples; they are of particular importance to our companion paper [27].

Theorem 1.1. *Let $f^*\colon X^* \to (\Delta^*)^k$ be a smooth family of PPAVs of dimension g, whose monodromy cone is matroidal (Defs. 2.7, 4.6). There is a flat, K-trivial extension $f\colon X \to \Delta^k$ which is a* nodal *morphism (Def. 5.1), and f may be assumed* strictly nodal *if the monodromy about each coordinate hyperplane is imprimitive.*

Moreover, given k generators of a matroidal cone, there exists a family $f^*\colon X^* \to (\Delta^*)^k$ of PPAVs whose monodromies are the specified generators, and an extension as above, which is the restriction of a projective family over a quasiprojective variety Y to a polydisk $\Delta^k \subset Y$.

See Theorem 7.1 and Corollary 7.2, respectively, for more algebraic formulations of the first and second statements of the above theorem.

As a particular application, the relative intermediate Jacobian fibration $IJY^\circ \to (\Delta^*)^{10}$ of the punctured universal deformation $Y^\circ \to (\Delta^*)^{10}$ of the Segre cubic Y_0 (Ex. 2.16) admits such a filling $IJY \to \Delta^{10} \simeq \mathrm{Def}_{Y_0}$ as does the relative Jacobian fibration of the universal deformation of a nodal curve C_0 (Exs. 2.15, 4.13). The respective matroids are the Seymour–Bixby matroid $\underline{R}_{10}$ and the cographic matroid $M^*(G)$ of the dual graph $G = \Gamma(C_0)$ of C_0 (Exs. 4.7, 4.3).

Nodal and semistable morphisms. Our investigations connect in a very natural way to the notion of a *semistable morphism* $f\colon X \to Y$, introduced by Abramovich–Karu [1] and the more restrictive notion of a *nodal morphism* (Def. 5.1). The former are morphisms between smooth spaces X and Y, which are étale or analytically-locally a product of snc degenerations $x^{(1)}\cdots x^{(m)} = u$, and the latter (nodal morphisms) additionally satisfy $m \leq 2$. The main result of Adiprasito–Liu–Temkin [2] is that, for *any* dominant morphism $f\colon X \to Y$, there is a regular alteration $Y' \to Y$ of the base and a birational modification of the base change $X' \to X \times_Y Y'$ so that $f'\colon X' \to Y'$ is semistable. Furthermore, once semistability is achieved, further base changes admit a functorial semistable resolution.

Once we have proven that transversely shifted matroidal degenerations are nodal morphisms, we continue with a general analysis of the Hodge theory of semistable morphisms, in Section 5. We define a multi-parameter analogue (Prop. 5.10) of the Clemens retraction for 1-parameter semistable degenerations, and investigate the relationship (Prop. 5.11) between the dual complex of the central fiber, and the graded piece $\mathrm{gr}_0^W V_{\mathbb{Z}}$ of the weight filtration on $V_{\mathbb{Z}} = H_1(X_t, \mathbb{Z})$.

In Section 5.3, we instantiate explicitly the functorial resolution of [2, Thm. 4.4], in the case of a base change of a nodal morphism (Thm. 5.14). We apply this resolution algorithm to transversely shifted matroidal degenerations in Section 5.4. Refinements of these results will play an important role in [27, Sec. 3 and 4].

The second Voronoi fan and Alexeev's theorem. Finally, we review the work of Alexeev, Nakamura, Namikawa and Faltings–Chai on the extension of the universal family $\mathcal{X}_g \to \mathcal{A}_g$ over the toroidal compactification $\overline{\mathcal{A}}_g^{\mathrm{vor}}$ associated to a distinguished fan $\mathfrak{F}_{\mathrm{vor}}$ (Defs. 6.1, 6.4), whose support is the cone $\mathcal{P}_g^+$ of positive semi-definite $g \times g$ matrices with rational null space.

We sketch a proof of Alexeev's theorem that $\overline{\mathcal{A}}_g^{\mathrm{vor}}$ is the normalization of the KSBA compactification of the moduli space of abelian torsors with theta divisor $(X, \epsilon\Theta)$, paying particular attention to the subtle differences between Alexeev's universal family $\mathcal{X}_g^\star \to \mathcal{A}_g$ and the universal family of abelian varieties $\mathcal{X}_g \to \mathcal{A}_g$ (Constr. 6.6 and Rem. 6.8).

We also provide a brief review of the extensive literature on the cones of the second Voronoi fan, for $g \leq 6$ (Ex. 6.7).

1.2. **Index of constructions.** The various forms of the Mumford construction presented in this paper are thus:

- (3.3): Via fans, over a 1-parameter base (i.e., a disk), and over a family of 1-parameter bases, complete with respect to a fixed monodromy operator $T\colon H_1(X_t,\mathbb{Z}) \to H_1(X_t,\mathbb{Z})$, encoded by a symmetric bilinear form $B \in \mathrm{Sym}^2(\mathrm{gr}_0^W V_{\mathbb{Z}})^\vee$.
- (3.8): Via fans, over a polydisk Δ^k, and over a family of such polydisks, complete with respect to a fixed collection of monodromy bilinear forms $B_i \in \mathrm{Sym}^2(\mathrm{gr}_0^W V_{\mathbb{Z}})^\vee$, $i = 1, \ldots, k$.
- (3.11): Via fans, over the toroidal extension $\mathcal{A}_g \hookrightarrow \mathcal{A}_g^{\mathbb{B}}$ associated to a rational polyhedral cone $\mathbb{B} = \mathbb{R}_{\geq 0}\{B_1, \ldots, B_k\} \subset \mathcal{P}_g^+$.
- (3.21): Via polytopes, over a polydisk Δ^k, associated to a collection $\{b_1, \ldots, b_k\}$ of convex piecewise linear functions $\mathbb{R}^g \to \mathbb{R}$ with appropriate $\mathbb{Z}^g$-periodicity, and over a family of polydisks, complete with respect to the associated monodromy cone $\mathbb{B}$.

(3.26): Via polytopes, over the toroidal extension $\mathcal{A}_g \hookrightarrow \mathcal{A}_g^{\mathbb{B}}$ associated to a rational polyhedral cone $\mathbb{B} = \mathbb{R}_{\geq 0}\{B_1, \ldots, B_k\} \subset \mathcal{P}_g^+$.

(3.38): Via polytopes, as in (3.21), but where only d times the principal polarization extends to a relatively ample line bundle on the family.

(4.16): As special cases of (3.21) and (3.38), associated to a regular matroid $\underline{R}$ of rank g, and a hyperplane arrangement inducing this regular matroid from the set of normal vectors.

(6.5): Associated to a "tautological" version of (3.26) for a cone $\mathbb{B} \in \mathfrak{F}_{\mathrm{vor}}$ of the second Voronoi fan, giving a local analytic extension of the universal family of abelian varieties.

(6.6): As in (6.5), but giving an extension $\overline{\mathcal{X}_g^\star}^{\mathrm{vor}} \to \overline{\mathcal{A}}_g^{\mathrm{vor}}$ of the universal family of abelian torsors with theta divisor.

An extensive collection of examples (Exs. 3.31, 3.34, 3.35, 3.40, 4.13, 4.14, 4.20, 4.21, 5.16, 6.7), with figures, is also provided in the text, see especially Section 3.5. The first of these (Ex. 3.31) is the prerequisite ur-example of the Mumford construction: the *Tate curve*, i.e. the extension of the family $\mathbb{C}^*/u^{\mathbb{Z}} \to \Delta_u^*$ of elliptic curves by an irreducible nodal curve.

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2. Preliminary material

2.1. **Algebraic and analytic stacks.** By a *DM algebraic stack*, or simply *DM stack*, we will mean a separated Deligne–Mumford stack of finite type over $\mathbb{C}$. Similarly, a *DM analytic stack* will be a separated Deligne–Mumford analytic stack X in the sense of [62, Def. 5.2]. Thus, X is a stack on the site of complex analytic spaces such that the diagonal is representable and finite and there exists an analytic space Y and a surjective étale morphism $Y \to X$. It follows that X is locally modeled as a finite quotient of an analytic space, see [62, Prop. 5.4], and that the analytification of a DM algebraic stack is a DM analytic stack, see [62, Lem. 5.5].

2.2. **Principally polarized abelian varieties.** Let $\mathcal{A}_g$ denote the DM stack of principally polarized abelian varieties (PPAVs) of dimension g, over $\mathbb{C}$. Since a PPAV X is uniquely determined by its polarized weight -1 Hodge structure on $H_1(X, \mathbb{Z})$, the period map defines an isomorphism $\mathcal{A}_g \simeq \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ to an arithmetic quotient of a Hermitian symmetric domain of Type III. We review this construction now.

Definition 2.1. A $\mathbb{Z}$-*polarized Hodge structure* $(V_{\mathbb{Z}}, H^{\bullet,\bullet}, L)$ *of weight* k is a $\mathbb{Z}$-module $V_{\mathbb{Z}}$ together with an integral, non-degenerate, $(-1)^k$-symmetric bilinear form $L\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}$, and a Hodge decomposition

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

satisfying the following conditions:

(1) $H^{q,p} = \overline{H^{p,q}}$ for all $p+q=k$,
(2) $L(H^{p,q}, H^{p',q'}) = 0$ unless $p = q'$, $q = p'$,
(3) $(-1)^{k(k-1)/2} i^{p-q} L(\bar{v}, v) > 0$ for all $0 \neq v \in H^{p,q}$.

Definition 2.2. A $\mathbb{Z}$-polarized Hodge structure $(V_{\mathbb{Z}}, H^{\bullet,\bullet}, L)$ is *principally polarized* if the pairing L is unimodular.

Let $(V_{\mathbb{Z}}, L)$ be a unimodular symplectic lattice, and consider the Lagrangian Grassmannian $\mathrm{LGr}(V_{\mathbb{C}}, L)$. It is the projective flag variety of isotropic g-dimensional subspaces of $V_{\mathbb{C}}$. The polarized weight -1 Hodge structures on $(V_{\mathbb{Z}}, L)$ with a Hodge decomposition of the form $V_{\mathbb{C}} = H^{-1,0} \oplus H^{0,-1}$ define an analytic open subset of $\mathrm{LGr}(V_{\mathbb{C}}, L)$, given by

$$\{H^{-1,0} \subset V_{\mathbb{C}} \,:\, L|_{H^{-1,0}} = 0 \text{ and } iL(\bar{v}, v) > 0 \text{ for } 0 \neq v \in H^{-1,0}\}. \tag{1}$$

Given $[H^{-1,0}]$ in (1), we may define a complex torus

$$X := V_{\mathbb{C}}/(V_{\mathbb{Z}} + H^{-1,0}).$$

We have canonical isomorphisms $H_1(X, \mathbb{Z}) \simeq V_{\mathbb{Z}}$ and $H^1(X, \mathbb{Z}) \simeq V_{\mathbb{Z}}^{\vee}$. Thus, the symplectic form $L \in V_{\mathbb{Z}}^{\vee} \wedge V_{\mathbb{Z}}^{\vee}$ defines an element

$$L \in \wedge^2 H^1(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}).$$

The condition that $H^{-1,0}$ is Lagrangian for L amounts to the property that $L \in H^{1,1}(X)$ is a Hodge class. Hence L determines a holomorphic line bundle $\mathcal{L} \to X$, unique up to translation by $\mathrm{Pic}^0(X)$. Finally, the condition $iL(\bar{v}, v) > 0$ ensures that any lift $\mathcal{L}$ is ample, and so in fact, X is an abelian variety (i.e. projective).

Choosing a standard symplectic basis of $V_{\mathbb{Z}}$ produces an isometry $(V_{\mathbb{Z}}, L) \simeq (\mathbb{Z}^{2g}, \cdot)$, where $\mathbb{Z}^{2g}$ is generated by vectors e_i, f_i for $i = 1, \dots, g$ and the unimodular symplectic form $\cdot$ satisfies $e_i \cdot e_j = f_i \cdot f_j = 0$ and $e_i \cdot f_j = \delta_{ij}$.

Definition 2.3. The *Siegel upper half-space* $\mathcal{H}_g$ is the space of symmetric $g \times g$ matrices with positive-definite imaginary part.

A choice of symplectic basis of $(V_{\mathbb{Z}}, L)$ identifies (1) with $\mathcal{H}_g$. In a standard symplectic basis, the Lagrangian $H^{-1,0} \subset V_{\mathbb{Z}} \otimes \mathbb{C}$ is the span of the columns of some $2g \times g$ *period matrix*

$$\begin{pmatrix} \sigma \\ I \end{pmatrix} \in \mathrm{Mat}_{2g \times g}(\mathbb{C}),$$

which we write in 2×1 block form. The condition that $L \in H^{1,1}(X)$ is Hodge amounts to the symmetry of σ, while the positivity condition $iL(\bar{v}, v) > 0$ amounts to the fact that the imaginary part $\mathrm{Im}(\sigma) > 0$ is positive-definite. Hence $\sigma \in \mathcal{H}_g$. Changes of symplectic basis, i.e. elements of $\mathrm{Sp}_{2g}(\mathbb{Z})$, act on the left, by 2×2 block matrices,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma \\ I \end{pmatrix} = \begin{pmatrix} A\sigma + B \\ C\sigma + D \end{pmatrix}.$$

Renormalizing the generators of our Lagrangian subspace to be dual to the e_i with respect to the symplectic form corresponds to right multiplication by $(C\sigma + D)^{-1}$. So we get the Lagrangian corresponding to $(A\sigma + B)(C\sigma + D)^{-1} \in \mathcal{H}_g$, which is the standard action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{H}_g$.

Definition 2.4. The pair (X, L) is called a *principally polarized abelian variety*, or PPAV.

For any representative $\mathcal{L} \in \mathrm{Pic}(X)$ of L, we have $h^0(X, \mathcal{L}) = 1$, and so there is a unique divisor $\Theta \in |\mathcal{L}|$ called the *theta divisor*.

It follows from the above discussion that the moduli stack of PPAVs (X, L) is given by the orbifold (i.e. smooth DM analytic stack) $\mathcal{A}_g \simeq \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g$. Furthermore, the universal family $\mathcal{X}_g \to \mathcal{A}_g$ of PPAVs is uniformized by $\mathbb{C}^g \times \mathcal{H}_g$ and can be presented as a quotient, too:

$$\mathcal{X}_g \simeq \left(\mathbb{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbb{Z})\right) \backslash (\mathbb{C}^g \times \mathcal{H}_g).$$

2.3. **Degenerations of PPAVs.** In the following sections, we discuss the monodromy of degenerations of PPAVs, especially in relation to toroidal extensions of $\mathcal{A}_g$. See [15] for reference.

Let $f\colon (X, L) \to \Delta^k$ be a degeneration of PPAVs of dimension g over a polydisk Δ^k with coordinates $u_1, \ldots, u_k$, such that the discriminant locus is the union of the coordinate hyperplanes $V(u_i) = \{u_i = 0\}$, for $i = 1, \ldots, k$. Fix a base point $t \in (\Delta^*)^k$ and let $V_{\mathbb{Z}} \coloneqq H_1(X_t, \mathbb{Z})$. Suppose that the monodromy transformation $T_i\colon V_{\mathbb{Z}} \to V_{\mathbb{Z}}$ associated to the simple, oriented loop $\gamma_i \in \pi_1((\Delta^*)^k, t) \simeq \mathbb{Z}^k$ is unipotent—for instance, by a result of Clemens [19, Thm. 7.36], this holds if the general fiber over $V(u_i)$ has reduced normal crossings.

Choosing a symplectic basis $(V_{\mathbb{Z}}, L) \simeq (\mathbb{Z}^{2g}, \cdot)$, we may view T_i as acting on the reference lattice $\mathbb{Z}^{2g}$. Let $N_i = \log(T_i) = T_i - I$ be its logarithm, where I denotes the identity matrix of size $2g \times 2g$. Note that $N_i^2 = 0$ and $N_i \circ N_j = N_j \circ N_i$ commute. Let $N = \sum_{i=1}^k r_i N_i$, $r_i \in \mathbb{N}$, be any strictly positive linear combination. Then N is the monodromy of the restriction of f to the cocharacter $\Delta \to \Delta^k$ defined by $u \mapsto (u^{r_1}, \ldots, u^{r_k})$. By [16, Thm. 3.3], all $(r_1, \ldots, r_k) \in \mathbb{N}^k$ define the same increasing *weight filtration*

$$\begin{aligned} W_{-2} &\coloneqq (\operatorname{im} N)^{\mathrm{sat}} \\ W_{-1} &\coloneqq \ker N \\ W_0 &\coloneqq V_{\mathbb{Z}}. \end{aligned}$$

More generally, for any $(r_1, \ldots, r_k) \in (\mathbb{Z}_{\geq 0})^k$, the filtration so defined depends only on the polyhedral face of $(\mathbb{R}_{\geq 0})^k$ containing N.

The above weight filtration may also be described as follows:

$$W_{-2} = \Big(\textstyle\sum_{i=1}^k \operatorname{im} N_i\Big)^{\mathrm{sat}} \quad \text{and} \quad W_{-1} = \textstyle\bigcap_{i=1}^k \ker(N_i). \tag{2}$$

Indeed, by the saturatedness of the above filtration, it suffices to prove this for rational, and, in fact, for real coefficients. The inclusions $\operatorname{im}(\sum N_i) \subset \sum \operatorname{im} N_i$ and $\bigcap \ker(N_i) \subset \ker(\sum N_i)$ are clear. To prove the converse, let $(r_1, \ldots, r_k) \in (\mathbb{R}_{>0})^k$ and note that $\operatorname{im}(\sum r_i N_i)$ and $\ker(\sum r_i N_i)$ do not depend on the choice of $r_i > 0$, see [16, Thm. 3.3]. The inclusions in question thus follow from a limit argument where $r_j = 1$ and $r_i \to 0$ for $i \neq j$, applied to $\operatorname{im}(\sum r_i N_i) \subset \operatorname{im}(\sum N_i)$ and $\ker(\sum N_i) \subset \ker(\sum r_i N_i)$.

Definition 2.5. Consider the standard Lagrangian subspace $\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_g \subset (\mathbb{Z}^{2g}, \cdot)$. Its stabilizer is the *parabolic group*

$$P_{\mathbb{Z}} := \left\{ \begin{pmatrix} A & B \\ 0 & A^{-T} \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}) \right\}$$

with $A \in \mathrm{GL}_g(\mathbb{Z})$ and $BA^T = AB^T$. We define the *unipotent subgroup* of $P_{\mathbb{Z}}$ to be

$$U_{\mathbb{Z}} := \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} : B \in \mathrm{Sym}_{g\times g}(\mathbb{Z}) \right\}$$

and the *Levi quotient* to be $P_{\mathbb{Z}}/U_{\mathbb{Z}} \simeq \mathrm{GL}_g(\mathbb{Z})$, which can also be lifted into $\mathrm{Sp}_{2g}(\mathbb{Z})$ as the block diagonal matrices (i.e. matrices with $B = 0$ in $P_{\mathbb{Z}}$).

The collection of commuting unipotent matrices $T_i \in \mathrm{Sp}_{2g}(\mathbb{Z})$ can be simultaneously conjugated into the unipotent subgroup $U_{\mathbb{Z}}$ as they fix a coisotropic space (given by W_{-1}) and hence fix a Lagrangian subspace. Thus, choosing a basis appropriately, we may assume that the monodromies T_i are all of the form

$$T_i = \begin{pmatrix} I & B_i \\ 0 & I \end{pmatrix} \tag{3}$$

for symmetric matrices B_i.

Definition 2.6. Let $f\colon (X, L) \to \Delta^k$ be a degeneration of PPAVs with unipotent monodromies about the coordinate hyperplanes. We define the *monodromy bilinear forms* $B_i \in \mathrm{Sym}^2(\mathrm{gr}_0^W V_{\mathbb{Z}})^\vee$ for $i = 1, \ldots, k$ by the formula

$$B_i(x, y) = L(N_i x, y). \tag{4}$$

Observe that B_i depends only on the punctured family $f^*\colon (X^*, L^*) \to (\Delta^*)^k$ and so the definition extends naturally to families of PPAVs over the punctured polydisk.

This provides a coordinate-free definition of the matrices B_i from above. Implicit in the above definition is the claim that N_i contains $W_{-1}V_{\mathbb{Z}}$ in its kernel, which follows from (2).

Definition 2.7. Choose a symplectic basis of $V_{\mathbb{Z}}$ such that T_i has the form (3) for each i, which identifies each B_i with a symmetric matrix $B_i \in \mathrm{Sym}_{g\times g}(\mathbb{Z})$. The span

$$\mathbb{B}_f \coloneqq \mathbb{R}_{\geq 0}\{B_1, \ldots, B_k\} \subset \mathrm{Sym}_{g\times g}(\mathbb{R})$$

is the *monodromy cone* associated to the degeneration $f\colon (X, L) \to \Delta^k$. This definition extends to any family of g-dimensional PPAVs $f^*\colon (X^*, L^*) \to (\Delta^*)^k$ with unipotent monodromy.

Note that the collection $(B_i)_{i=1,\ldots,k}$ of monodromy matrices, and hence the monodromy cone $\mathbb{B}_f$, is unique, up to the simultaneous action of $A \in \mathrm{GL}_g(\mathbb{Z})$ by $B_i \mapsto AB_iA^T$. This action corresponds to the conjugation action $T_i \mapsto MT_iM^{-1}$ of $P_{\mathbb{Z}} \subset \mathrm{Sp}_{2g}(\mathbb{Z})$, $M \in P_{\mathbb{Z}}$, which descends to the Levi quotient $P_{\mathbb{Z}}/U_{\mathbb{Z}} \simeq \mathrm{GL}_g(\mathbb{Z})$ because $U_{\mathbb{Z}}$ is commutative.

The symplectic basis of $V_{\mathbb{Z}}$ determines a lift of the classifying map $\Phi\colon (\Delta^*)^k \to \mathcal{A}_g$ to a holomorphic period map $\widetilde{\Phi}\colon \mathbb{H}^k \to \mathcal{H}_g$, where $\mathbb{H}^k \to (\Delta^*)^k$ is the universal cover, $\mathbb{H} \coloneqq \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Take coordinates $(\tau_1, \ldots, \tau_k) \in \mathbb{H}^k$, with the universal covering map given by $u_i = \exp(2\pi i \tau_i)$. This lifted map satisfies the equivariance property

$$\widetilde{\Phi}(\tau_1, \ldots, \tau_i + 1, \ldots, \tau_k) = T_i \cdot \widetilde{\Phi}(\tau_1, \ldots, \tau_k) = \widetilde{\Phi}(\tau_1, \ldots, \tau_k) + B_i$$

corresponding to the deck transformation for the generator $\gamma_i \in \pi_1((\Delta^*)^k, t)$.

Definition 2.8. Define a holomorphic map to the flag variety $\mathbb{D}^\vee \coloneqq \mathrm{LGr}(V_{\mathbb{Z}} \otimes \mathbb{C}, L)$ by

$$\begin{aligned}\widetilde{\Psi}\colon \mathbb{H}^m &\to \mathbb{D}^\vee \\ \tau &\mapsto \widetilde{\Phi}(\tau) - (\tau_1 B_1 + \cdots + \tau_k B_k)\end{aligned}$$

and denote by $\Psi\colon (\Delta^*)^k \to \mathbb{D}^\vee$ its descent to $(\Delta^*)^k$.

Note that $\widetilde{\Psi}$ descends because the $-(\tau_1 B_1 + \cdots + \tau_k B_k)$ term cancels the equivariance of $\widetilde{\Phi}$, and so makes the map invariant under the action of $\mathbb{Z}^k$. We now recall Schmid's multivariable nilpotent orbit theorem [57, Thm. 4.12], applied to our setting:

Theorem 2.9. *Ψ extends to a holomorphic map $\Delta^k \to \mathbb{D}^\vee$. Let $\Psi(0)$ denote the extension to the origin, and consider $\widetilde{\Phi}_{\mathrm{nilp}}(\tau) \coloneqq \Psi(0) + (\tau_1 B_1 + \cdots + \tau_k B_k)$. Then*

(1) $\widetilde{\Phi}_{\mathrm{nilp}}(\tau) \in \mathcal{H}_g \subset \mathbb{D}^\vee$ for all sufficiently large $\mathrm{Im}\,\tau_i$ and
(2) the distance $d(\widetilde{\Phi}(\tau), \widetilde{\Phi}_{\mathrm{nilp}}(\tau))$ decays exponentially in $\mathrm{Im}\,\tau_i$.

Here distance is measured in the natural left $\mathrm{Sp}_{2g}(\mathbb{R})$-invariant metric on $\mathcal{H}_g$.

Definition 2.10. Let $\mathcal{P}_g \coloneqq \{B \in \mathrm{Sym}_{g\times g}(\mathbb{R}) \,:\, B > 0\}$ be the cone of positive-definite matrices and let $\mathcal{P}_g^+$ be its *rational closure*, consisting of all positive semi-definite matrices whose kernel is a rational subspace of $\mathbb{R}^g$.

There is a natural stratification

$$\mathcal{P}_g^+ = \mathcal{P}_g \sqcup \textstyle\bigsqcup_{V_1} \mathcal{P}_{g-1} \sqcup \bigsqcup_{V_2} \mathcal{P}_{g-2} \sqcup \cdots \sqcup \mathcal{P}_0$$

where the $V_i \subset \mathbb{Z}^g$ range over all primitive integral sublattices of $\mathbb{Z}^g$ of codimension i, and the relevant copy of $\mathcal{P}_{g-i}$ is the cone of positive-definite bilinear forms on $\mathbb{R}^g/(V_i \otimes \mathbb{R})$. See Figure 22 to visualize the projectivization of $\mathcal{P}_2^+$, which is a cusped hyperbolic disk.

It follows from item (1) of Theorem 2.9 that:

Corollary 2.11. *The monodromy cone $\mathbb{B}_f$ is contained in the rational closure $\mathcal{P}_g^+$ of the positive-definite $g \times g$ matrices, uniquely up to the action of $\mathrm{GL}_g(\mathbb{Z})$.*

We have that $W_{-2}^{\perp} = W_{-1}$ where the perpendicular is taken with respect to L. Thus L descends to a unimodular symplectic form on W_{-1}/W_{-2}. In fact, by [57, Thm. 6.16], we have the following fundamental theorem on the existence of the limit MHS:

Theorem 2.12. *The tuple $(V_{\mathbb{Z}}, L, \Psi(0), W_\bullet)$ defines a graded-polarized $\mathbb{Z}$-mixed Hodge structure, the* limit mixed Hodge structure. *In particular, the filtration of $W_{-1}/W_{-2} \otimes \mathbb{C}$ induced by the Lagrangian subspace $\Psi(0)$ defines a pure, principally polarized Hodge structure of weight -1 on $W_{-1}/W_{-2} \simeq (\mathbb{Z}^{2h}, \cdot)$. Here h is the rank of the null space of a general element of $\mathbb{B}_f$.*

Definition 2.13. We say that $f\colon X \to \Delta^k$ is *maximally degenerate* if $\mathbb{B}_f \cap \mathcal{P}_g \neq \emptyset$, i.e. the general element of $\mathbb{B}_f$ is positive-definite. Equivalently, $W_{-2} = W_{-1}$, i.e. $h = 0$.

Definition 2.14. We define the *tropical moduli space of abelian varieties* to be

$$(\mathcal{A}_g)_{\mathrm{trop}} := \mathrm{GL}_g(\mathbb{Z})\backslash\mathcal{P}_g^+$$

where the action is via $B \mapsto ABA^T$.

We can view $(\mathcal{A}_g)_{\mathrm{trop}}$ as the tropical moduli space of abelian varieties, because, as we will see in Section 2.5, a fan defining a toroidal extension of $\mathcal{A}_g$ lives naturally in $(\mathcal{A}_g)_{\mathrm{trop}}$. But more deeply, $(\mathcal{A}_g)_{\mathrm{trop}}$ (or at least, the image of $\mathcal{P}_g$ in it) is itself a moduli space of "tropical abelian varieties" [42, 18, 14]: It parametrizes isometry classes of full rank lattices in $\mathbb{R}^g$, where the action of $\mathrm{GL}_g(\mathbb{Z})$ serves the role of forgetting the basis of the lattice in which the Gram matrix of the corresponding real intersection form has been expanded.

Then $\mathbb{B}_f$ defines, canonically, an immersed polyhedral cone in $(\mathcal{A}_g)_{\mathrm{trop}}$.

Example 2.15 (Degenerations of Jacobians)**.** Let $\pi\colon C \to \Delta^k$ be a family of nodal curves, which is smooth over the complement of the coordinate hyperplanes $V(u_1 \cdots u_k)$.

Let $\{p_{ij}\}$ denote the nodes of the general fiber over $V(u_i)$. The local equation of the smoothing of the node p_{ij} is given by $x_{ij}y_{ij} = u_i^{r_{ij}}$ for some positive integer r_{ij}. It follows from the Picard-Lefschetz formula that the logarithm of monodromy on $V_{\mathbb{Z}} = H_1(C_t, \mathbb{Z})$ about the i-th coordinate hyperplane is given by

$$N_i\colon x \mapsto -\textstyle\sum_j r_{ij}(x \cdot \gamma_{ij})\gamma_{ij} \tag{5}$$

where $\gamma_{ij} \in H_1(C_t, \mathbb{Z})$ is (either orientation of) the vanishing cycle of the node p_{ij} and $\cdot$ is the intersection form on $V_{\mathbb{Z}}$. Observe that the total space C is smooth if and only if all $r_{ij} = 1$ and no node of C_0 is the specialization of a node over both $V(u_i)$ and $V(u_j)$ for $i \neq j$. Taking the relative Jacobian fibration

$$J\pi^\circ\colon JC^\circ \to (\Delta^*)^k$$

of the smooth family, the monodromy is given by the same formula, since $H_1(JC_u, \mathbb{Z}) \simeq H_1(C_u, \mathbb{Z})$ for $u \in (\Delta^*)^k$. The weight filtration is

$$W_{-2} = \mathbb{Z}\text{-span}\{\gamma_{ij}\} \text{ and } W_{-1} = (W_{-2})^\perp.$$

Computing the monodromy bilinear forms on $\mathrm{gr}_0^W V_{\mathbb{Z}}$ we get

$$B_i(x,y) := N_i(x) \cdot y = \textstyle\sum_j r_{ij}(x \cdot \gamma_{ij})(y \cdot \gamma_{ij}).$$

Suppose now that $\pi\colon C \to \Delta^k \subset \mathrm{Def}(C_0)$ is a slice of the universal deformation of C_0 which is transversal to the equisingular/locally trivial deformations. Then there is only one node over each $V(u_i)$, the corresponding integer $r_i = 1$, and k is the number of nodes of C_0. Thus,

$$B_i(x,x) = (\gamma_i \cdot x)^2 \in \mathrm{Sym}^2(\mathrm{gr}_0^W V_{\mathbb{Z}})^\vee$$

is a rank 1 quadratic form, given by the square of the linear form which is pairing with the vanishing cycle $\gamma_i \in \mathrm{gr}_{-2}^W V_{\mathbb{Z}}$.

We have canonical isomorphisms

$$\begin{aligned}\mathrm{gr}_0^W V_{\mathbb{Z}} &\simeq H_1(\Gamma(C_0), \mathbb{Z}),\\ \mathrm{gr}_{-2}^W V_{\mathbb{Z}} &\simeq H^1(\Gamma(C_0), \mathbb{Z}),\end{aligned}$$

where $\Gamma(C_0)$ is the dual graph of the central fiber; indeed, by duality it suffices to prove the first isomorphism, which follows e.g. from Proposition 5.12 below. The space $W_{-2}V_{\mathbb{Z}} = \mathrm{gr}_{-2}^W V_{\mathbb{Z}}$ is generated by the vanishing cycles γ_i which are in bijection with the edges of $\Gamma(C_0)$. The relations between the vanishing cycles γ_i are given by the boundaries of the subsurfaces they bound. In terms of graphs, these are the coboundaries of the vertices of $\Gamma(C_0)$, so that

$$\mathrm{gr}_{-2}^W V_{\mathbb{Z}} \simeq \mathrm{coker}(\mathcal{C}^0(\Gamma(C_0), \mathbb{Z}) \xrightarrow{\partial} \mathcal{C}^1(\Gamma(C_0), \mathbb{Z})) =: H^1(\Gamma(C_0), \mathbb{Z}).$$

In turn, $H_1(\Gamma(C_0), \mathbb{Z}) \simeq \mathrm{gr}_0^W V_{\mathbb{Z}}$ with the quadratic form $B_i = (\gamma_i \cdot x)^2$ evaluating on a 1-cycle $\sum c_i e_i \in H_1(\Gamma(C_0), \mathbb{Z})$ to the square c_i^2 of the coefficient of the edge i in it. See Figure 1.

It follows from the Mayer–Vietoris sequence that $\mathrm{gr}_{-1}^W V_{\mathbb{Z}} \simeq H_1(C_0^\nu, \mathbb{Z})$ where $C_0^\nu \to C_0$ is the normalization, with its natural polarized $\mathbb{Z}$-Hodge structure.

Example 2.16 (Degenerations of intermediate Jacobians)**.** This example is due to Gwena [33]. Let Y_0 be the unique cubic threefold with 10 isolated A_1 singularities, the *Segre cubic.* Concretely, it is defined by the equations

$$Y_0 := \left\{\textstyle\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0\right\} \subset \mathbb{P}^5$$

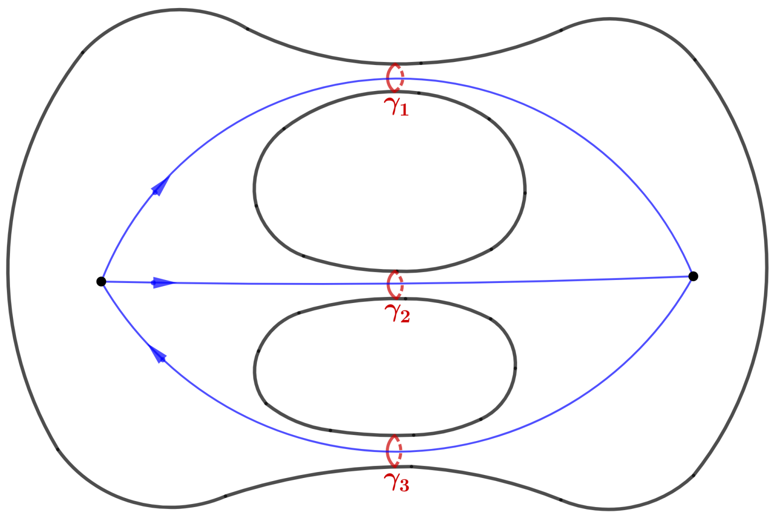


FIGURE 1. Nearby fiber C_t to the universal deformation $\pi\colon C \to \Delta^3$ of a nodal genus 2 curve C_0 with three nodes. Vanishing cycles $\{\gamma_1, \gamma_2, \gamma_3\} \in \mathrm{gr}^W_{-2} V_{\mathbb{Z}}$ shown in red, and dual graph $\Gamma(C_0)$ of the central fiber, shown in blue.

with the 10 singularities given by the S_6-orbit of the point $[1 : 1 : 1 : -1 : -1 : -1] \in \mathbb{P}^5$. Then, the universal deformation (whose existence follows from [29, Cor. 4.2], for example)

$$\pi\colon Y \to \mathrm{Def}_{Y_0} \simeq \Delta^{10}$$

of Y_0 is a degeneration of smooth cubic threefolds, whose discriminant locus is the union of the 10 coordinate hyperplanes $V(u_1 \cdots u_{10}) \subset \Delta^{10}$. Similar to the universal deformation of a nodal curve, if we label the nodes of Y_0 and coordinate hyperplanes appropriately, then $V(u_i) \simeq \Delta^9 \subset \Delta^{10}$ is exactly the locus where the i-th node is not smoothed.

We now define the *intermediate Jacobian fibration*

$$IJ\pi^\circ\colon IJY^\circ \to (\Delta^*)^{10}.$$

The rank 10, polarized, unimodular $\mathbb{Z}$-local system $\mathbb{V}_{\mathbb{Z}} := (R^3\pi^\circ_*\mathbb{Z}(1))^\vee$ underlies a variation of Hodge structure of weight -1 and type $(-1, 0)$, $(0, -1)$. It is polarized by the negation of the intersection form on $H_3(Y_u, \mathbb{Z})$ for $u \in (\Delta^*)^{10}$. Then, we define $IJY^\circ := \mathbb{V}_{\mathbb{C}}/(\mathcal{F}^0 + \mathbb{V}_{\mathbb{Z}})$.

Let $p_i \in Y_i$ be the unique node on the general fiber of $Y_i \to V(u_i)$. Then associated to p_i is the cycle of a vanishing 3-sphere $\gamma_i \in V_{\mathbb{Z}} = H_3(Y_t, \mathbb{Z}(-1))$. With an appropriate labeling and orientations, the 10 cycles γ_i satisfy the following collection of linear relations

$$\begin{aligned}\gamma_6 &= \gamma_5 - \gamma_1 + \gamma_2\\ \gamma_7 &= \gamma_1 - \gamma_2 + \gamma_3\\ \gamma_8 &= \gamma_2 - \gamma_3 + \gamma_4\\ \gamma_9 &= \gamma_3 - \gamma_4 + \gamma_5\\ \gamma_{10} &= \gamma_4 - \gamma_5 + \gamma_1,\end{aligned} \tag{6}$$

see [33, Sec. 7.1.10], and generate a primitive integral Lagrangian subspace $W_{-2} \subset H_3(Y_t, \mathbb{Z}(-1))$ of rank 5. The Dehn twist about the three-sphere γ_i gives the formula

$$N_i \colon x \mapsto -(x \cdot \gamma_i)\gamma_i$$

for the logarithm of monodromy about the i-th coordinate axis. By the same computation as Example 2.15, the monodromy bilinear forms on $\mathrm{gr}_0^W V_{\mathbb{Z}}$ satisfy $B_i(x,x) = (\gamma_i \cdot x)^2$. As we will see in Remark 4.8 in Section 4.1, there is no graph G for which the γ_i correspond to the edges of G and such that the relations (6) between the γ_i are given by the image of the map $C^0(\Gamma(G), \mathbb{Z}) \to C^1(\Gamma(G), \mathbb{Z})$ defined by some choice of orientation, cf. Example 2.15.

We will give explicit extensions of the families $J\pi^\circ$ and $IJ\pi^\circ$ over Δ^k and Δ^{10} in Examples 4.13, 4.14 and in Construction 4.16.

2.4. **Toric varieties.** We recall here some of the basic theory of toric varieties. We refer to [30, 21] for the standard notions.

Toric geometry will be used both to extend $\mathcal{A}_g$ toroidally (Section 2.5), and to build Mumford degenerations (Section 3). We employ the standard toric notations of $\mathbf{N} \simeq \mathbb{Z}^g$ for a free abelian group of rank g and $\mathbf{M} \coloneqq \mathrm{Hom}(\mathbf{N}, \mathbb{Z})$, for constructions concerning the abelian and degenerate abelian fibers. These lattices play, respectively, the roles of the cocharacter and the character lattices in toric geometry. Thus, fans lie in $\mathbf{N}$ while polytopes lie in $\mathbf{M}$.

Definition 2.17. A *fan* $\mathfrak{F}$ in $\mathbf{N}$ is a set of strongly convex, rational polyhedral cones $\tau \subset \mathbf{N}_{\mathbb{R}}$ for which every face of a cone is a cone, and the intersection of two cones is a face of each.

We do not impose the hypothesis that fans have finitely many cones, and indeed almost none of the fans in this paper satisfy this hypothesis. For each cone $\tau \in \mathfrak{F}$ in a fan, we may consider the $\mathbb{C}$-algebra $\mathbb{C}[\tau^\vee \cap \mathbf{M}]$ associated to the commutative semi-group $\tau^\vee \cap \mathbf{M}$, where $\tau^\vee \subset \mathbf{M}_{\mathbb{R}}$ is the collection of all linear functionals evaluating non-negatively on $\tau \subset \mathbf{N}_{\mathbb{R}}$. We form the corresponding affine toric variety

$$Y(\tau) \coloneqq \mathrm{Spec}\, \mathbb{C}[\tau^\vee \cap \mathbf{M}]. \tag{7}$$

Then the gluing $Y(\mathfrak{F}) \coloneqq \bigcup_{\tau \in \mathfrak{F}} Y(\tau)$ of these affine schemes along the natural open immersions corresponding face inclusions gives the *toric variety* $Y(\mathfrak{F})$, see [30, Sec. 1.4].

Notation 2.18. If $\mathfrak{F}$ is a polyhedral fan, we notate its toric variety by $Y(\mathfrak{F})$.

For all constructions in this paper, $Y(\mathfrak{F})$ will be an analytic space (and even a $\mathbb{C}$-scheme) which is locally of finite type.

As in the usual theory of toric varieties, the torus orbits of dimension d in $Y(\mathfrak{F})$, isomorphic necessarily to $(\mathbb{C}^*)^d$, correspond bijectively to cones of codimension d in $\mathfrak{F}$, see [30, Sec. 3.1].

Definition 2.19. A *polytope* H is a convex set in $\mathbf{M}_\mathbb{R}$ defined by the intersection of a (possibly infinite) number of closed rational half-spaces, such that H is *locally of finite type*, i.e. locally about every point $p \in H$, it is defined by a finite number of half-spaces.

Definition 2.20. A *face* $F \subset H$ of a polytope is a non-empty intersection of H with a (possibly empty) collection of supporting hyperplanes, and the *local monoid* $\mathbf{M}_F$ of this face is the intersection of the lattice $\mathbf{M}$ with the finitely many (possibly empty) closed half-spaces which define H in the neighborhood of a general point of F, translated to the origin.

The *normal fan* of H is the fan formed from the dual cones of the local monoids $\mathbf{M}_F$ ranging over all faces $F \subset H$ (including the open face H).

The toric variety $Y = Y_H$ associated to the polytope H is the union of $\operatorname{Spec}\mathbb{C}[\mathbf{M}_F]$ ranging over all faces, see also [30, Sec. 1.5]. If, furthermore, all faces of H are integral polytopes, then there is a canonically defined torus-equivariant holomorphic line bundle $\mathcal{L}$ on Y, given by gluing together line bundles on each affine chart $\operatorname{Spec}\mathbb{C}[\mathbf{M}_F]$, in a manner which locally agrees with the recipe in [30, Sec. 3].

Remark 2.21. When the polytope H is compact, we may construct $Y = Y_H$ directly as the projective variety $Y_H = \operatorname{Proj}\mathbb{C}[\operatorname{Cone}(H) \cap (\mathbf{M} \times \mathbb{Z})]$, where $\operatorname{Cone}(H)$ is the cone over H, put at height 1 in $\mathbf{M}_\mathbb{R} \times \{1\} \subset \mathbf{M}_\mathbb{R} \times \mathbb{R}$; the above line bundle is $\mathcal{L} = \mathcal{O}_Y(1)$. The lattice points $\mathbf{m} \in H \cap \mathbf{M}$ define a basis of torus-equivariant sections of $H^0(Y, \mathcal{O}(1))$. More generally, lattice points $(\mathbf{m}, w) \in \operatorname{Cone}(H) \cap (\mathbf{M} \times \{w\})$ of height w correspond to torus-equivariant sections $\theta_{\mathbf{m}/w} \in H^0(Y, \mathcal{O}(w))$. The multiplication map

$$H^0(Y, \mathcal{O}(w_1)) \otimes H^0(Y, \mathcal{O}(w_2)) \to H^0(Y, \mathcal{O}(w_1 + w_2))$$

corresponds to $((\mathbf{m}, w_1), (\mathbf{m}', w_2)) \mapsto (\mathbf{m} + \mathbf{m}', w_1 + w_2)$, which we may equivalently write as a multiplication rule $\theta_{\mathbf{m}/w_1} \cdot \theta_{\mathbf{m}'/w_2} = \theta_{(\mathbf{m}+\mathbf{m}')/(w_1+w_2)}$, cf. (19) below.

Suppose now that there is a subgroup $A \subset \operatorname{GL}(\mathbf{M}) \ltimes \mathbf{M}$ of the integral-affine group (possibly infinite), acting on the polytope H. Then there is a natural action of A on Y, which in a basis $\mathbf{M} \simeq \mathbb{Z}^g$ acts on the open torus orbit $\mathbf{N} \otimes \mathbb{C}^* \simeq (\mathbb{C}^*)^g$ by the map

$$c_i \mapsto c_1^{a_{i1}} \cdots c_g^{a_{ig}}$$

where $(a_{ij})_{1 \le i,j \le g} \in \operatorname{GL}_g(\mathbb{Z})$ is the linear part of A in the chosen basis. Moreover, there is a linearization of $\mathcal{L} = \mathcal{O}_Y(1)$ with respect to the A-action on Y, which acts on sections by

$a \cdot (z^m) = z^{a \cdot m}$. The A-action is properly discontinuous on a tubular neighborhood of all toric strata corresponding to faces F, or cones $\tau \in \mathfrak{F}$, whose A-stabilizer has finite order.

Definition 2.22. A fan $\mathfrak{F}$ is *regular* if every cone $\tau \in \mathfrak{F}$ is *standard affine*; that is, the primitive integral generators of the extremal rays of τ form a subset of a basis of $\mathbf{N}$.

Equivalently, if $\mathfrak{F}$ is a normal fan, the polytope H should be *Delzant*. The toric variety $Y_H = Y(\mathfrak{F})$ is a smooth analytic space if and only if $\mathfrak{F}$ is regular, in which case for each $\tau \in \mathfrak{F}$, the affine toric variety $Y(\tau)$, see (7), is isomorphic to a product of an affine space with a torus.

Definition 2.23. The *support* (in $\mathbf{N}_{\mathbb{R}}$) of a fan $\mathfrak{F}$ is the union of all cones $\tau \in \mathfrak{F}$.

Finally, a *morphism* of fans $\mathfrak{F} \to \mathfrak{G}$ is a linear map between the corresponding cocharacter lattices, which sends cones into cones. It induces a torus-equivariant map $Y(\mathfrak{F}) \to Y(\mathfrak{G})$.

2.5. **Toroidal extensions.** We now outline the construction of toroidal extensions of $\mathcal{A}_g$ associated to a monodromy cone $\mathbb{B}_f$ and more generally a fan $\mathfrak{F}$. See [51, 9, 45, 17] for references on toroidal compactifications of Siegel spaces.

Definition 2.24. A *fan* $\mathfrak{F}$ for $\mathcal{A}_g$ is a rational polyhedral decomposition whose support is contained in $\mathcal{P}_g^+$, for which $\mathfrak{F}$ is $\mathrm{GL}_g(\mathbb{Z})$-invariant under the action $B \mapsto ABA^T$, with finitely many orbits of cones.

Example 2.25. Let $f\colon X \to \Delta^k$ be an abelian fibration with unipotent monodromies about the coordinate hyperplanes. Then $\mathfrak{F}_f := \mathrm{GL}_g(\mathbb{Z}) \cdot \mathbb{B}_f$ defines a fan, when $\mathbb{B}_f$ injects into $(\mathcal{A}_g)_{\mathrm{trop}} = \mathrm{GL}_g(\mathbb{Z}) \backslash \mathcal{P}_g^+$.

Let $\mathbb{B}$ be a polyhedral cone which embeds into $(\mathcal{A}_g)_{\mathrm{trop}}$. In what follows, we will consider the fan $\mathfrak{F} = \mathrm{GL}_g(\mathbb{Z}) \cdot \mathbb{B}$. Consider the coordinate-wise exponential mapping:

$$\begin{aligned} E\colon \mathrm{Sym}_{g\times g}(\mathbb{C}) &\to \mathrm{Sym}_{g\times g}(\mathbb{C}^*) \\ (\sigma_{ij})_{i,j=1}^g &\mapsto (\exp(2\pi i \sigma_{ij}))_{i,j=1}^g. \end{aligned}$$

The map E is the quotient by the action of translation by $U_{\mathbb{Z}} \simeq \mathrm{Sym}_{g\times g}(\mathbb{Z})$, so E defines an open embedding of the quotient of Siegel space (Def. 2.3) into a torus

$$\overline{E}\colon U_{\mathbb{Z}} \backslash \mathcal{H}_g \hookrightarrow U_{\mathbb{Z}} \backslash \mathrm{Sym}_{g\times g}(\mathbb{C}) \simeq \mathrm{Sym}_{g\times g}(\mathbb{C}^*) \simeq \mathrm{Sym}_{g\times g}(\mathbb{Z}) \otimes \mathbb{C}^* \simeq U_{\mathbb{Z}} \otimes \mathbb{C}^*.$$

Here $U_{\mathbb{Z}} \subset P_{\mathbb{Z}}$ is the unipotent radical of the parabolic, as in Definition 2.5, and the isomorphism $\mathrm{Sym}_{g\times g}(\mathbb{Z}) \otimes \mathbb{C}^* \simeq \mathrm{Sym}_{g\times g}(\mathbb{C}^*)$ is given by $(n_{ij})_{1\le i,j\le g} \otimes \lambda = (\lambda^{n_{ij}})_{1\le i,j\le g}$. This is called the *first* or *unipotent partial quotient* of $\mathcal{H}_g$ in the theory of toroidal compactifications.

Compose the period maps $\widetilde{\Phi}$ and $\widetilde{\Phi}_{\mathrm{nilp}}$ of any degeneration with $\mathbb{B}_f = \mathbb{B}$ with the quotient map $\mathcal{H}_g \to \mathrm{Sym}_{g\times g}(\mathbb{Z}) \backslash \mathcal{H}_g$. They descend to single-valued maps $\Phi, \Phi_{\mathrm{nilp}}\colon (\Delta^*)^k \to \mathrm{Sym}_{g\times g}(\mathbb{Z}) \backslash \mathcal{H}_g$. Composing Φ_{nilp} with the map $\overline{E}$ gives rise to a map

$$(\Delta^*)^k \to \mathrm{Sym}_{g\times g}(\mathbb{Z}) \otimes \mathbb{C}^* \simeq \mathrm{Sym}_{g\times g}(\mathbb{C}^*) \tag{8}$$

whose image is (an analytic open subset of) a translate of the subtorus $\langle\mathbb{B}\rangle\otimes\mathbb{C}^*$ where $\langle\mathbb{B}\rangle := (\mathbb{R}\mathbb{B})\cap\mathrm{Sym}_{g\times g}(\mathbb{Z})$. Thus, Theorem 2.9 can be rephrased as saying that the period mapping is approximated by a translate of a subtorus, near $0\in\Delta^k$.

Since the fan $\mathfrak{F}=\mathrm{GL}_g(\mathbb{Z})\cdot\mathbb{B}$ sits in the co-character lattice $\mathrm{Sym}_{g\times g}(\mathbb{Z})$, the associated toric variety $Y(\mathfrak{F})$ is a toroidal extension of $\mathrm{Sym}_{g\times g}(\mathbb{C}^*)$. Consider the quotient $\mathrm{GL}_g(\mathbb{Z})\backslash Y(\mathfrak{F})$. This quotient is not globally well-behaved, e.g. the action of $\mathrm{GL}_g(\mathbb{Z})$ fixes the origin point of the open torus orbit. But there is a tubular analytic neighborhood $T(\mathfrak{F})\subset Y(\mathfrak{F})$ of the union of the toric boundary strata of $Y(\mathfrak{F})$ corresponding to cones intersecting $\mathcal{P}_g$ on which the $\mathrm{GL}_g(\mathbb{Z})$-action is properly discontinuous. Let $T^*(\mathfrak{F})$ be the intersection of $T(\mathfrak{F})$ with the open torus orbit $U_\mathbb{Z}\otimes\mathbb{C}^*$. Then, we have open embeddings

$$\mathrm{GL}_g(\mathbb{Z})\backslash T(\mathfrak{F}) \hookleftarrow \mathrm{GL}_g(\mathbb{Z})\backslash T^*(\mathfrak{F}) \hookrightarrow P_\mathbb{Z}\backslash\mathcal{H}_g. \tag{9}$$

By [11, Thm. 4.9(iv)], the boundary of $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ is locally modeled near the Baily-Borel cusp (associated to the Lagrangian subspace $\mathbb{Z}e_1\oplus\cdots\oplus\mathbb{Z}e_g$) by $P_\mathbb{Z}\backslash\mathcal{H}_g$. Thus, by (9), we may glue $\mathrm{GL}_g(\mathbb{Z})\backslash T(\mathfrak{F})$ to $\mathcal{A}_g=\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ along their common analytic open subset $\mathrm{GL}_g(\mathbb{Z})\backslash T^*(\mathfrak{F})$.

More generally, the same construction applies to any $\mathrm{GL}_g(\mathbb{Z})$-invariant fan $\mathfrak{F}$, and the resulting toroidal extension is relatively proper over the 0-dimensional cusp of the Baily-Borel [11] compactification $\overline{\mathcal{A}}_g^{\mathrm{BB}}=\mathcal{A}_g\sqcup\mathcal{A}_{g-1}\sqcup\cdots\sqcup\mathcal{A}_1\sqcup\mathcal{A}_0$ if and only if $\mathrm{Supp}(\mathfrak{F})=\mathcal{P}_g^+$.

When $\mathfrak{F}$ contains cones supported in $\mathcal{P}_g^+\setminus\mathcal{P}_g$, the gluing defined by (9) further extends along the intermediate-dimensional strata of the Baily-Borel compactification. This is the *toroidal extension* $\mathcal{A}_g\hookrightarrow\mathcal{A}_g^{\mathfrak{F}}$ of the orbifold $\mathcal{A}_g$.

When $\mathrm{Supp}(\mathfrak{F})=\mathcal{P}_g^+$, we notate the toroidal extension by $\mathcal{A}_g\hookrightarrow\overline{\mathcal{A}}_g^{\mathfrak{F}}$; it is proper, and we call it a *toroidal compactification*. It follows from [9, Thm. 5.2] that $\mathcal{A}_g^{\mathfrak{F}}$ or $\overline{\mathcal{A}}_g^{\mathfrak{F}}$ is a DM algebraic stack, in the former case by refining and extending $\mathfrak{F}$ to a fan with full support $\mathcal{P}_g^+$.

Notation 2.26. For simplicity, we will write $\mathcal{A}_g\hookrightarrow\mathcal{A}_g^{\mathbb{B}}$ to notate the toroidal extension associated to the fan $\mathfrak{F}=\mathrm{GL}_g(\mathbb{Z})\cdot\mathbb{B}$ consisting of the orbit of a polyhedral cone $\mathbb{B}\subset\mathcal{P}_g^+$.

Proposition 2.27. *For any degeneration of PPAVs $f^*\colon X^*\to(\Delta^*)^k$ with monodromy cone $\mathbb{B}$, the period map $(\Delta^*)^k\to\mathcal{A}_g$ admits a unique holomorphic extension $\Delta^k\to\mathcal{A}_g^{\mathbb{B}}$.*

Proof. The proposition follows from Theorem 2.9: We have shown that the nilpotent orbit $\Phi_{\rm nilp}$ maps, analytically-locally near the boundary of $\mathcal{A}_g$, into the translate of the subtorus $\langle\mathbb{B}\rangle\otimes\mathbb{C}^*\subset\mathrm{Sym}_{g\times g}(\mathbb{C}^*)$, see (8). Any cocharacter $B\otimes(\mathbb{C}\setminus 0)\subset\langle\mathbb{B}\rangle\otimes\mathbb{C}^*$, for $B\in\mathbb{B}$, admits a completion over $0\in\mathbb{C}$ to the toroidal extension $\mathcal{A}_g^{\mathbb{B}}$, sending 0 into the toroidal boundary stratum corresponding to the cone of $\mathfrak{F}$ containing B in its relative interior. We deduce that $\Phi_{\rm nilp}\colon(\Delta^*)^k\to\mathcal{A}_g$ extends holomorphically to a map $\Delta^k\to\mathcal{A}_g^{\mathbb{B}}$.

Next, it follows from the exponential convergence (2) of Φ towards $\Phi_{\rm nilp}$ (in the invariant metric on $\mathcal{H}_g$) that $\Phi\colon(\Delta^*)^k\to\mathcal{A}_g$ admits a continuous extension $\Delta^k\to\mathcal{A}_g^{\mathbb{B}}$. Since Δ^k is normal, Riemann's removable singularities theorem implies that this continuous extension is holomorphic. It is unique because the toroidal extension is a separated analytic stack. □

Remark 2.28. In general, one may wish to consider monodromy cones $\mathbb{B}_f$ for which the $\mathbb{B}_f$ does not embed into $(\mathcal{A}_g)_{\mathrm{trop}}$ (Def. 2.14). The issue here is that the $\mathrm{GL}_g(\mathbb{Z})$-orbit of such a cone may intersect itself. This problem is resolved by rather quotienting $U_{\mathbb{Z}}\backslash\mathcal{H}_g \subset U_{\mathbb{Z}} \otimes \mathbb{C}^*$ by a finite index subgroup $\Gamma \subset \mathrm{GL}_g(\mathbb{Z})$.

For instance, consider the n-torsion subgroup $X[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ of a PPAV X. The principal polarization defines a natural non-degenerate symplectic pairing on $X[n]$. We define the moduli space of abelian varieties X with *full Lagrangian level n structure* by adding the data of:

(1) a Lagrangian subspace $\Xi \simeq (\mathbb{Z}/n\mathbb{Z})^g \subset X[n]$ and
(2) a $\mathbb{Z}/n\mathbb{Z}$-basis of Ξ.

All degenerations we consider in this paper admit some full Lagrangian level n structure, because there is a distinguished Lagrangian subspace $\mathrm{gr}^W_{-2}H_1(X,\mathbb{Z})$ on which the monodromy acts trivially. The moduli stack $\mathcal{A}_g[n]$ of abelian varieties with full Lagrangian level n structure is an étale cover $\widetilde{\mathcal{A}}_g = \mathcal{A}_g[n] \to \mathcal{A}_g$.

At an appropriate 0-cusp of the Baily–Borel compactification of $\widetilde{\mathcal{A}}_g$ (the cusp corresponding to a Lagrangian subspace $\widetilde{\Xi} \subset H_1(X,\mathbb{Z})$ for which $\frac{1}{n}\widetilde{\Xi}/\widetilde{\Xi} = \Xi$), the parabolic stabilizer $P_{\mathbb{Z}}$ has the following structure:

$$0 \to U_{\mathbb{Z}} \to P_{\mathbb{Z}} \to \Gamma(n) \to 0,$$

where the unipotent subgroup $U_{\mathbb{Z}} \simeq \mathrm{Sym}_{g\times g}(\mathbb{Z})$ is the same as without level structure, but the Levi quotient $\Gamma(n) := \ker(\mathrm{GL}_g(\mathbb{Z}) \to \mathrm{GL}_g(\mathbb{Z}/n\mathbb{Z}))$ is the full level n subgroup.

Then, a toroidal extension of $\widetilde{\mathcal{A}}_g$ at this Baily–Borel cusp has the "advantage" that a fan need only be $\Gamma(n)$-invariant. Furthermore, the fundamental domain for the action of $\Gamma(n)$ is larger than the fundamental domain for that action of $\mathrm{GL}_g(\mathbb{Z})$. In particular, given any polyhedral cone $\mathbb{B} \subset \mathcal{P}_g^+$, there exists some n (depending on $\mathbb{B}$) for which $\mathbb{B}$ embeds into the quotient $\Gamma(n)\backslash\mathcal{P}_g^+$. The preceding results also apply at this cusp, since $\Gamma(n)\cdot\mathbb{B}$ now defines a fan.

Proposition 2.29. *Let $\tau \subset \mathbb{R}^k$ be a strictly convex, rational polyhedral cone, and consider any homomorphism $\phi\colon \mathbb{Z}^k \to \mathrm{Sym}_{g\times g}(\mathbb{Z})$ for which $\mathbb{B} := \phi_{\mathbb{R}}(\tau) \subset \mathcal{P}_g^+$. There exists a quasiprojective variety Y, divisor $D \subset Y$, point $0 \in D$, and projective family $f^*\colon X^* \to Y^*$, $Y^* = Y \setminus D$, of PPAVs of dimension g, in the algebraic category, whose monodromy cone at 0 is given by ϕ in the following sense:*

(1) Y admits, near $0 \in Y$, an étale-local isomorphism to the toric variety $Y(\tau)$, sending D to the toric boundary, and 0 to the torus fixed point.
(2) the monodromy representation of $Y \setminus D$ near 0 is given, under this isomorphism, by

$$\pi_1(\mathbb{Z}^k \otimes \mathbb{C}^*, *) \simeq \mathbb{Z}^k \xrightarrow{\phi} \mathrm{Sym}_{g\times g}(\mathbb{Z}) = U_{\mathbb{Z}} \subset \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Here $\mathbb{Z}^k \otimes \mathbb{C}^* \subset Y(\tau)$ is the open torus orbit.

Observe that it follows from (1) that Y^* is smooth near 0.

Proof. We first prove the proposition under the hypothesis that ϕ is injective.

Consider the étale cover $\mathcal{A}_g[n] \to \mathcal{A}_g$ given by full Lagrangian level $n \geq 3$ structure as in Remark 2.28. As noted, there is a Baily–Borel cusp of $\mathcal{A}_g[n]$ whose parabolic stabilizer $P_{\mathbb{Z}}$ has unipotent subgroup $U_{\mathbb{Z}} \subset \mathrm{Sym}_{g\times g}(\mathbb{Z})$ which is the same as for $\mathcal{A}_g$ but whose Levi quotient is $P_{\mathbb{Z}}/U_{\mathbb{Z}} = \Gamma(n) \subset \mathrm{GL}_g(\mathbb{Z})$. Since $\Gamma(n)$ is neat for $n \geq 3$, its action on the toroidal extension $\mathcal{A}_g[n]^{\mathfrak{F}}$ is free in the tubular neighborhood $T(\mathfrak{F})$ of (9). We deduce that there is a Zariski open $V \subset \mathcal{A}_g[n]^{\mathfrak{F}}$, containing all maximally degenerate strata, over which the universal family $\mathcal{X}_g|_{V^*} \to V^*$ of PPAVs, $V^* := V \cap \mathcal{A}_g[n]$, exists as a scheme (rather than just a DM stack).

Then, choose $n \geq 3$ so that $\mathbb{B}$ lies in the strict interior of a fundamental domain for the action of $\Gamma(n)$ on $\mathcal{P}_g^+$ and define $\mathfrak{F} = \Gamma(n) \cdot \mathbb{B}$. By the construction of toroidal extensions, the open set V is étale-locally isomorphic to the toric variety $Y_{\mathrm{Sym}_{g\times g}(\mathbb{R})}(\mathbb{B})$.

Choose a finite index sublattice $\Lambda \subset \mathrm{Sym}_{g\times g}(\mathbb{Z})$ for which $\Lambda \cap \mathbb{B} = \mathrm{im}(\phi)$. Then the finite étale cover $\Lambda \otimes \mathbb{C}^* \to \mathrm{Sym}_{g\times g}(\mathbb{Z}) \otimes \mathbb{C}^*$ of algebraic tori induces a branched cover of toric varieties $Y_{\Lambda\otimes\mathbb{R}}(\tau) \to Y_{\mathrm{Sym}_{g\times g}(\mathbb{R})}(\mathbb{B})$. Take an algebraic branched cover $V' \to V$ with the same branching over the toric boundary, under the étale-local identification of $\mathcal{A}_g[n]^{\mathfrak{F}}$ with $Y_{\mathrm{Sym}_{g\times g}(\mathbb{R})}(\mathbb{B})$. Then, V' is étale-locally isomorphic to $Y_{\Lambda\otimes\mathbb{R}}(\tau)$, over the deepest toroidal stratum.

Finally, to construct Y, we slice V' by $\binom{g+1}{2} - \dim \tau$ generic hyperplanes, which under the étale-local identification of V' with $Y_{\Lambda\otimes\mathbb{R}}(\tau)$ are transversal to the deepest toroidal boundary stratum. We set Y^* to be the inverse image of V^*, $D = Y \setminus Y^*$, and $0 \in D$ as an intersection point of the hyperplanes with the deepest stratum. We set the family of abelian varieties $f^*\colon X^* \to Y^*$ to be the pullback of the universal family $\mathcal{X}_g|_{V^*} \to V^*$ (which exists by the above discussion) along the map $Y^* \to V^*$.

To verify that the monodromy representation of f^* is as specified, consider a 1-parameter family $\Delta^* \to Y^*$ which extends to a map $\Delta \to Y$ sending $0 \mapsto 0$. The monodromy over Δ^* can be computed in the étale-local model as the monodromy of the family of PPAVs over a co-character. Such co-characters correspond to lattice points, in $\tau \cap \mathbb{Z}^k$. Since $f^*\colon X^* \to Y^*$ is pulled back from V^*, the monodromy representation is pulled back along the morphism of cocharacter lattices, given by the inclusion $\Lambda \hookrightarrow \mathrm{Sym}_{g\times g}(\mathbb{Z})$ which, in particular, restricts $\phi\colon \mathbb{Z}^k \to \mathrm{Sym}_{g\times g}(\mathbb{Z})$. Finally, it suffices to observe that the monodromy over the cocharacter $B \otimes \mathbb{C}^* \subset \mathrm{Sym}_{g\times g}(\mathbb{Z}) \otimes \mathbb{C}^*$ is canonically identified with $B \in \mathcal{P}_g^+ \cap \mathrm{Sym}_{g\times g}(\mathbb{Z})$, see e.g. (11).

Finally, we address the case where ϕ is not injective. Define $\overline{\tau} := \phi_{\mathbb{R}}(\tau) \subset \mathcal{P}_g^+$, viewed as a polyhedral cone in the lattice $\phi(\mathbb{Z}^k)$. We have a descended, injective homomorphism $\overline{\phi}\colon \phi(\mathbb{Z}^k) \hookrightarrow \mathrm{Sym}_{g\times g}(\mathbb{Z})$ satisfying the hypotheses of the proposition. Thus, there is a family $\overline{f}^*\colon \overline{X}^* \to \overline{Y}^*$ over a base $\overline{Y}^* = \overline{Y} \setminus \overline{D}$, a point $\overline{0} \in \overline{D}$, and an étale-local isomorphism $\alpha\colon \overline{Y} \to Y(\overline{\tau})$ near $\overline{0}$ satisfying the conclusion of the proposition. Then the base change of the morphism $\alpha \circ \overline{f}^*\colon \overline{X}^* \to Y(\overline{\tau})$ along the morphism of toric varieties $Y(\tau) \to Y(\overline{\tau})$ produces the desired family $f^*\colon X^* \to Y^* := \overline{Y}^* \times_{Y(\overline{\tau})} Y(\tau) \subset \overline{Y} \times_{Y(\overline{\tau})} Y(\tau) =: Y$. Here we take $0 \in Y$ to be the fiber product of the points $\overline{0} \in \overline{Y}$ and the torus fixed point of $Y(\tau)$. Since the desired monodromy map factors through $\overline{\phi}$, the condition on monodromy follows. $\square$

Corollary 2.30. *Let $B_i \in \mathcal{P}_g^+ \cap \mathrm{Sym}_{g\times g}(\mathbb{Z})$ for $i = 1, \ldots, k$. There is a smooth quasiprojective variety Y, an snc divisor $D \subset Y$, a zero-stratum $0 \in D$, and a projective algebraic family $f^*\colon X^* \to Y^*$ of PPAVs, such that the local monodromy bilinear form (Def. 2.6) about the component $D_i \ni 0$ is B_i for all $i = 1, \ldots, k$.*

Note that B_i need not be primitive.

Proof. The corollary follows from Proposition 2.29, by taking $\tau := \mathbb{R}^k_{\geq 0} \subset \mathbb{R}^k$, and the homomorphism $\phi\colon \mathbb{Z}^k \to \mathrm{Sym}_{g\times g}(\mathbb{Z})$ sending $e_i \mapsto B_i$. Here one can assume that Y is smooth and D is snc because they are étale-locally isomorphic to $Y(\tau) = \mathbb{C}^k$ and the union of the coordinate hyperplanes, respectively. $\square$

Remark 2.31. Given any two families X_1^*, X_2^* of PPAVs over $(\Delta^*)^k$, with the same integral monodromies about each coordinate axis, there is an analytic deformation $\mathcal{X}^* \to (\Delta^*)^k \times Z$ of such families, over a connected base Z, and points $1, 2 \in Z$ for which $\mathcal{X}_i^* \simeq X_i^*$ for $i = 1, 2$. Indeed, we may first deform each X_i^* to the nilpotent orbit (8) passing through the same point of the deepest toroidal stratum of $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$, and then relate the two translates of subtori $\langle \mathbb{B} \rangle \otimes \mathbb{C}^*$ by a translation in $\mathrm{Sym}_{g\times g}(\mathbb{C}^*)$.

3. The Mumford construction

In Section 2.5, we have shown how to extend $\mathcal{A}_g \to \mathcal{A}_g^{\mathbb{B}}$ (or $\widetilde{\mathcal{A}}_g \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$) toroidally, so that the period map $(\Delta^*)^k \to \mathcal{A}_g$ of any degeneration $f\colon X \to \Delta^k$ with monodromy cone $\mathbb{B}_f = \mathbb{B}$ admits an extension of the period map over the punctures $\Delta^k \to \mathcal{A}_g^{\mathfrak{F}}$. We will now describe how to extend the universal family of PPAVs over $\mathcal{A}_g$ so that we may pull back this extension, to produce particularly nice birational models of degenerations. It is useful to have examples in mind; many are provided in Section 3.5. All degenerations we consider in this section are maximal, in the sense of Definition 2.13.

3.1. Mumford construction, fan version.

Let $B \in \mathrm{Sym}^2 \mathbf{M}^\vee$ be a positive-definite, symmetric, integral, bilinear form on a lattice $\mathbf{M} \simeq \mathbb{Z}^g$. Then B defines a homomorphism

$$\begin{aligned} N\colon \mathbf{M} &\to \mathbf{M}^\vee = \mathbf{N}, \\ \mathbf{m} &\mapsto B(\mathbf{m}, -). \end{aligned}$$

Define $\Lambda_B \subset \mathbf{N}$ to be $\mathrm{im}(N)$. In terms of symmetric $g \times g$ matrices, Λ_B is the span of the rows (or columns) of B, and so defines a finite index sublattice of $\mathbf{N} \simeq \mathbb{Z}^g$.

Definition 3.1. $X_{\mathrm{trop}}(B) := \mathbf{N}_{\mathbb{R}}/\Lambda_B$ is the *tropical abelian variety* associated to B.

Definition 3.2. A (resp. $\mathbb{Q}$-)*tiling* of $X_{\mathrm{trop}}(B) = \mathbf{N}_{\mathbb{R}}/\Lambda_B$ is a decomposition into convex polytopes with integer (resp. rational) vertices, or equivalently, a Λ_B-periodic (resp. $\mathbb{Q}$-)polytopal tesselation of $\mathbf{N}_{\mathbb{R}}$. A *complete triangulation* of $X_{\mathrm{trop}}(B)$ is a tiling, all of whose polytopes are lattice simplices of minimal volume $(1/g!)$.

Construction 3.3 (1-parameter case)**.** Let $B \in \mathrm{Sym}^2\mathbf{M}^\vee$ be positive-definite and let $\mathcal{T}$ be a $\mathbb{Q}$-tiling of $X_{\mathrm{trop}}(B)$. We define the 1-parameter Mumford degeneration associated to $\mathcal{T}$.

Embed $\mathbf{N}_{\mathbb{R}} \simeq \mathbf{N}_{\mathbb{R}} \times \{1\} \hookrightarrow \mathbf{N}_{\mathbb{R}} \times \mathbb{R}$ as an affine hyperplane at height 1 in a space of one dimension higher. Then, the cone over the tiling $\mathrm{Cone}(\mathcal{T})$ defines a rational polyhedral fan in $\mathbf{N}_{\mathbb{R}} \times \mathbb{R} \simeq \mathbb{R}^{g+1}$. See Figure 8 for an example of a tiling $\mathcal{T}$ of $\mathbb{R}^2/\Lambda_B$, where

$$B = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}.$$

Let $Y(\mathrm{Cone}(\mathcal{T}))$ denote the corresponding infinite type toric variety. The action of Λ_B by translations on $\mathcal{T}$ lifts to a linear action $\Lambda_B \hookrightarrow \mathrm{GL}_{g+1}(\mathbb{Z})$ on $\mathbf{N}_{\mathbb{R}} \times \mathbb{R}$ acting on the fan $\mathrm{Cone}(\mathcal{T})$, and hence induces an action of Λ_B on $Y(\mathrm{Cone}(\mathcal{T}))$ by automorphisms.

Observe that the height function $\mathbf{N}_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$, given by projecting to the final coordinate, defines a morphism of fans $\mathrm{Cone}(\mathcal{T}) \to \mathbb{R}_{\geq 0}$ to the fan of $\mathbb{C}$ (which is simply the positive ray in $\mathbb{R}$). Hence, there is an induced map of toric varieties

$$Y(\mathrm{Cone}(\mathcal{T})) \to Y(\mathbb{R}_{\geq 0}) = \mathbb{C}.$$

Since the action of Λ_B preserves the height function, this morphism descends to the quotient $\Lambda_B \backslash Y(\mathrm{Cone}(\mathcal{T})) \to \mathbb{C}$, though this full quotient is poorly behaved. Let u be the monomial coordinate about $0 \in \mathbb{C}$. By standard toric geometry, we have:

(1) The fiber of $Y(\mathrm{Cone}(\mathcal{T})) \to \mathbb{C}$ over $u \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is $\mathbf{N} \otimes \mathbb{C}^* \simeq (\mathbb{C}^*)^g$, and the fiber over $u = 0 \in \mathbb{C}$ (i.e. the toric boundary) is an infinite quilt of complete toric varieties, whose dual complex is the original tiling $\mathcal{T}$.

(2) Λ_B acts on the dual complex $\mathcal{T}$ of the toric boundary by translation, and acts on the fiber over $u \in \mathbb{C}^*$ by translations by the rank g subgroup $u^B := \Lambda_B \otimes u \subset \mathbf{N} \otimes \mathbb{C}^*$ which in coordinates is the subgroup

$$\langle (u^{B_{11}}, \ldots, u^{B_{1g}}),\, \ldots,\, (u^{B_{g1}}, \ldots, u^{B_{gg}}) \rangle \subset (\mathbb{C}^*)^g, \quad B = (B_{ij}) \in \mathrm{Sym}_{g\times g}(\mathbb{Z}).$$

Then, since B is positive-definite, $u^B \subset \mathbf{N} \otimes \mathbb{C}^*$ is a discrete subgroup for all $u \in \Delta^*$. Hence the action of $\mathbf{M} \simeq \Lambda_B$ is properly discontinuous over the unit disk Δ, because the action is clearly properly discontinuous over $u = 0$ since $\mathbf{M}$ acts freely on the tiles of $\mathcal{T}$. Setting $X(\mathrm{Cone}(\mathcal{T})) := \Lambda_B \backslash Y(\mathrm{Cone}(\mathcal{T}))^{u \in \Delta}$, we get a proper, complex-analytic degeneration

$$f \colon X(\mathrm{Cone}(\mathcal{T})) \to \Delta$$

of complex tori, called the *standard* 1*-parameter Mumford degeneration* associated to the $\mathbb{Q}$-tiling $\mathcal{T}$ of $X_{\mathrm{trop}}(B) = \mathbf{N}_{\mathbb{R}}/\Lambda_B$.

The general fiber is principally polarized, by a symplectic form L defined as follows: Let $u \in \Delta^*$. Then, noting that we have a canonical isomorphism $H_1(\mathbf{N} \otimes \mathbb{C}^*, \mathbb{Z}) \simeq \mathbf{N}$, we have a canonical exact sequence

$$0 \to \mathbf{N} \to H_1(X_u, \mathbb{Z}) \xrightarrow{\sigma} \mathbf{M} \to 0 \tag{10}$$

induced by the long exact sequence of homotopy groups associated to the fibration

$$u^B \hookrightarrow \mathbf{N} \otimes \mathbb{C}^* \to \mathbf{N} \otimes \mathbb{C}^*/u^B = X_u.$$

Here we use that we have canonical isomorphisms $u^B \simeq \Lambda_B \simeq \mathbf{M}$. Then, we may define a unimodular symplectic form on $H_1(X_u, \mathbb{Z})$ by choosing a splitting of σ and then using the canonical pairing between $\mathbf{M}$ and $\mathbf{N}$. The splitting of σ we choose is specified by a choice of logarithm $2\pi i\tau = \log u$. Such a choice gives a presentation

$$X_u = \mathbf{N}_{\mathbb{C}}/(\mathbf{N} \oplus \tau N(\mathbf{M})) \simeq \mathbb{C}^g/(\mathbb{Z}^g \oplus \tau B(\mathbb{Z}^g));$$

the resulting symplectic form L is then independent of the choice of logarithm τ because the transformation $\tau \mapsto \tau+1$ defines a symplectomorphism of $H_1(X_u, \mathbb{Z})$. Concretely, the symplectic form L is given as $L((\mathbf{n}, \mathbf{m}), (\mathbf{n}', \mathbf{m}')) = \mathbf{m}'(\mathbf{n}) - \mathbf{m}(\mathbf{n}')$, for $(\mathbf{n}, \mathbf{m}), (\mathbf{n}', \mathbf{m}') \in \mathbf{M} \oplus \mathbf{N}$.

More generally, given a symmetric $g \times g$ matrix $a = (a_{ij}) \in \mathrm{Sym}_{g\times g}(\mathbb{C}^*)$, we may perform the same construction, but instead quotient by the subgroup

$$au^B := \langle (a_{11}u^{B_{11}}, \ldots, a_{1g}u^{B_{1g}}),\ \ldots,\ (a_{g1}u^{B_{g1}}, \ldots, a_{gg}u^{B_{gg}}) \rangle.$$

We must take $|u|$ small enough that $-B \log |u| - \log |a_{ij}| > 0$. Then this construction may be performed relatively, choosing a in some subtorus of $\mathrm{Sym}_{g\times g}(\mathbb{C}^*)$ forming coset representatives for the natural action of $u \in \mathbb{C}^*$. We get an analytic degeneration

$$f_\circ^{\mathrm{univ}} \colon X_\circ^{\mathrm{univ}}(\mathrm{Cone}(\mathcal{T})) \to \Delta^{\mathrm{univ}}$$

where $\Delta^{\mathrm{univ}} \to \mathrm{Sym}_{g\times g}(\mathbb{C}^*)/\mathbb{C}^*$ is a holomorphic disk bundle over $\mathrm{Sym}_{g\times g}(\mathbb{C}^*)/\mathbb{C}^*$, such that $f_\circ^{\mathrm{univ}}$ restricts to a family of PPAVs over the punctured disc bundle $(\Delta^*)^{\mathrm{univ}} \subset \Delta^{\mathrm{univ}}$.

In terms of the toroidal extensions of Section 2.5, Δ^{univ} maps to the toroidal extension $\mathcal{A}_g^{\mathfrak{F}}$ whose fan is (the orbit of) a single ray $\mathfrak{F} = \mathrm{GL}_g(\mathbb{Z}) \cdot \mathbb{R}_{\geq 0} B$. So every 1-parameter degeneration of PPAVs with monodromy B admits an extension pulled back from $f_\circ^{\mathrm{univ}}$ along an arc transverse to the boundary divisor $\{0\} \times \mathrm{Sym}_{g\times g}(\mathbb{C}^*)/\mathbb{C}^* = \Delta^{\mathrm{univ}} \setminus (\Delta^*)^{\mathrm{univ}}$. ♣

Remark 3.4. Allowing the tiling $\mathcal{T}$ to have strictly rational vertices corresponds to allowing the components of the central fiber to have non-reduced components. More precisely, the irreducible components $V_i \subset X_0(\mathrm{Cone}(\mathcal{T})) := f^{-1}(0)$ are in bijection with the 0-cells $v_i \in \mathcal{T}$, and the multiplicity d_i of V_i is the smallest positive integer for which $d_i v_i \in \mathbf{N}$.

Remark 3.5. When $\mathcal{T}$ has integral vertices, the total space $X(\mathrm{Cone}(\mathcal{T}))$ or $X_\circ^{\mathrm{univ}}(\mathrm{Cone}(\mathcal{T}))$ of the Mumford degeneration is smooth if and only if $\mathcal{T}$ is a complete triangulation, i.e. $\mathcal{T}$ is a tiling by standard lattice simplices. This condition ensures that $\mathrm{Cone}(\mathcal{T})$ is a regular fan—its cones are all standard affine cones. Then $X(\mathrm{Cone}(\mathcal{T})) \to \Delta$ is a semistable, K-trivial degeneration (with smooth total space), see Proposition 3.14 below. These are sometimes called Kulikov models, in analogy to K3 degenerations.

Remark 3.6. By passing to an intermediate cover $\mathcal{H}_g \to \widetilde{\mathcal{A}}_g \to \mathcal{A}_g$ as in Remark 2.28, we may assume that the Levi quotient $\Gamma \subset \mathrm{GL}_g(\mathbb{Z})$ of the parabolic stabilizer at some Baily–Borel 0-cusp of $\widetilde{\mathcal{A}}_g$ acts freely on $\mathcal{P}_g$. Then, when B is primitive, the boundary divisor of $\widetilde{\mathcal{A}}_g^{\mathbb{R}_{\geq 0}B}$ is isomorphic to the discriminant divisor of $f_\circ^{\mathrm{univ}}$, rather than a further finite quotient of it. In this case, the base of $f_\circ^{\mathrm{univ}}$ glues onto $\widetilde{\mathcal{A}}_g$ and thus, we may extend $f_\circ^{\mathrm{univ}}$ to a family $f^{\mathrm{univ}}\colon X^{\mathrm{univ}}(\mathrm{Cone}(\mathcal{T})) \to \widetilde{\mathcal{A}}_g^{\mathbb{R}_{\geq 0}B}$ extending the universal family over $\widetilde{\mathcal{A}}_g$. A priori, this extension exists only in the category of analytic spaces. But it is always an algebraic space (Prop. 6.12), and for certain choices of tiling $\mathcal{T}$, we may ensure it is relatively projective, see Section 3.3.

When B is not primitive, the base Δ^{univ} of $f_\circ^{\mathrm{univ}}$ rather maps to $\widetilde{\mathcal{A}}_g^{\mathbb{R}_{\geq 0}B}$ by a map ramified over the toroidal boundary divisor, to order the imprimitivity of B.

Proposition 3.7. *The standard* 1*-parameter Mumford degeneration* $f\colon X(\mathrm{Cone}(\mathcal{T})) \to \Delta$ *corresponding to a tiling* $\mathcal{T}$ *of* $\mathbf{N}_\mathbb{R}/\Lambda_B$ *has monodromy invariant* B*. The period map on the universal cover of* Δ^* *is the nilpotent orbit through the origin: The maps*

$$\widetilde{\Phi}, \widetilde{\Phi}_{\mathrm{nilp}}\colon \mathbb{H} \to \mathrm{Sym}_{g\times g}(\mathbb{Z})\backslash\mathcal{H}_g$$

satisfy $\widetilde{\Phi}(\tau) = \widetilde{\Phi}_{\mathrm{nilp}}(\tau) = \tau B$*, where* $u = e^{2\pi i\tau}$ *is the coordinate on* Δ^**. More generally, the period map on the restriction of* $f_\circ^{\mathrm{univ}}$ *to the universal cover of* $(\Delta^*)^{\mathrm{univ}}|_{\{a\}}$ *for* $a \in \mathrm{Sym}_{g\times g}(\mathbb{C}^*)$*, is given by* $\widetilde{\Phi}(\tau) = \widetilde{\Phi}_{\mathrm{nilp}}(\tau) = \frac{1}{2\pi i}(\log a_{ij}) + \tau B$*.*

Proof. On the one hand, the coordinatewise exponential $E(\tau B)$ is given by

$$\begin{pmatrix} \exp(2\pi i B_{11}\tau) & \cdots & \exp(2\pi i B_{1g}\tau) \\ \cdots & & \cdots \\ \exp(2\pi i B_{g1}\tau) & \cdots & \exp(2\pi i B_{gg}\tau) \end{pmatrix} = \begin{pmatrix} u^{B_{11}} & \cdots & u^{B_{1g}} \\ \cdots & & \cdots \\ u^{B_{g1}} & \cdots & u^{B_{gg}} \end{pmatrix}. \tag{11}$$

On the other hand, the fiber of the universal family $\mathcal{X}_g \to \mathcal{A}_g$ over the period matrix $\sigma \in \mathcal{H}_g$ is simply the complex torus $\mathbb{C}^g/(\mathbb{Z}^g \oplus \mathbb{Z}^g\sigma) \simeq (\mathbb{C}^*)^g/\langle \text{rows of } E(\sigma)\rangle$ so the proposition follows. ☐

Construction 3.8 (k-parameter case)**.** We now extend Construction 3.3 to the multivariable setting. Consider a collection of positive semi-definite bilinear forms $B_i \in \mathrm{Sym}^2\mathbf{M}^\vee$ for $i = 1, \ldots, k$ for which $\sum_{i=1}^k B_i$ is positive-definite. These bilinear forms define a collection of symmetric homomorphisms $N_i\colon \mathbf{M} \to \mathbf{N}$.

We consider the quotient of $\mathbf{N} \otimes \mathbb{C}^* \simeq (\mathbb{C}^*)^g$ by the subgroup

$$\begin{aligned} u_1^{B_1}\cdots u_k^{B_k} \coloneqq \langle (u_1^{(B_1)_{11}}\cdots u_k^{(B_r)_{11}}, \ldots, u_1^{(B_1)_{1g}}\cdots u_k^{(B_k)_{1g}}), \ldots, \\ (u_1^{(B_1)_{g1}}\cdots u_k^{(B_k)_{g1}}, \ldots, u_1^{(B_1)_{gg}}\cdots u_k^{(B_k)_{gg}})\rangle \simeq \mathbb{Z}^g. \end{aligned}$$

If $|u_i| < 1$ for all i, the resulting action of $u_1^{B_1} \cdots u_k^{B_k}$ on $(\mathbb{C}^*)^g$ is properly discontinuous. Thus, the quotient is a fibration

$$f^*\colon X^*(B_1, \ldots, B_k) \to (\Delta^*)^k \tag{12}$$

of PPAVs, over a punctured polydisk, where the base has coordinates u_i.

To define an extension over Δ^k, we require a fan $\mathcal{S}$ inside of $\mathbf{N}_\mathbb{R} \times \mathbb{R}^k \simeq \mathbb{R}^{g+k}$ which is $\mathbf{M}$-periodic for an action respecting the projection to $\mathbb{R}^k$. More precisely, declare $\mathbf{m} \in \mathbf{M}$ to act linearly on $\mathbf{N}_\mathbb{R} \times \mathbb{R}^k$ by

$$(\mathbf{n}, \vec{r}) \mapsto (\mathbf{n} + (\vec{r} \cdot \vec{N})(\mathbf{m}), \vec{r}) \in \mathbf{N}_\mathbb{R} \times \mathbb{R}^k, \tag{13}$$

where $\vec{r} \cdot \vec{N} := r_1 N_1 + \cdots + r_k N_k$ and $N_i\colon \mathbf{M} \to \mathbf{N}$ are the symmetric homomorphisms associated to $B_i \in \mathrm{Sym}^2 \mathbf{M}^\vee$ as above.

The fan $\mathcal{S}$ must then be $\mathbf{M}$-periodic (with respect to the action (13)) and the projection to $\mathbb{R}^k$ must induce a morphism of fans to $(\mathbb{R}_{\geq 0})^k$, such that $\mathrm{Supp}(\mathcal{S})$ contains $\mathbb{R}^g \times (1, \ldots, 1)$. Furthermore, we usually require that the morphism $\mathcal{S} \to (\mathbb{R}_{\geq 0})^k$ is *flat*, that is, the image of any cone of $\mathcal{S}$ is a cone of $(\mathbb{R}_{\geq 0})^k$.

Then the multivariable Mumford construction is the result of quotienting by $\mathbf{M}$ the inverse image $Y(\mathcal{S})^{u \in \Delta^k}$ of Δ^k in the infinite type toric variety $Y(\mathcal{S})$. We call this quotient

$$f\colon X(\mathcal{S}) \to \Delta^k. \tag{14}$$

It is a proper, analytic, flat extension of $f^*\colon X^*(B_1, \ldots, B_k) \to (\Delta^*)^k$, with flatness guaranteed by the flatness of the fan map. As in Construction 3.3, the fibration $\mathbf{M} = u_1^{B_1} \cdots u_k^{B_k} \hookrightarrow \mathbf{N} \otimes \mathbb{C}^* \to X_u := f^{-1}(u)$ defines an exact sequence

$$0 \to \mathbf{N} \to H_1(X_u, \mathbb{Z}) \xrightarrow{\sigma} \mathbf{M} \to 0, \qquad u = (u_1, \ldots, u_k) \in (\Delta^*)^k, \tag{15}$$

and by choosing a section of σ by taking logarithms of u_i, the canonical pairing between $\mathbf{M}$ and $\mathbf{N}$ induces a well-defined principal polarization on X_u.

Over the co-character $\Delta \to \Delta^k$ defined by $u \mapsto (u^{r_1}, \ldots, u^{r_k})$, the construction specializes to the 1-variable Mumford Construction 3.3 associated to $B = r_1 B_1 + \cdots + r_k B_k$ and the relevant fan is the restriction of $\mathcal{S}$ to the inverse image of $\mathbb{R}_{\geq 0}\vec{r} \subset (\mathbb{R}_{\geq 0})^k$; here $\mathcal{S}|_{\mathbb{R}_{\geq 0}\vec{r}} \simeq \mathrm{Cone}(\mathcal{T})$ for a tiling $\mathcal{T}$ depending on $\vec{r}$. In particular, by Proposition 3.7, the monodromy cone (see Definition 2.7) of the degeneration $f^*\colon X^*(B_1, \ldots, B_k) \to (\Delta^*)^k$ is given by $\mathbb{B} := \mathbb{R}_{\geq 0}\{B_1, \ldots, B_k\}$.

More generally, we may, as in Construction 3.3, perform the multivariable construction relatively over the torus $(\mathrm{Sym}_{g\times g}(\mathbb{Z})/\langle \mathbb{B} \rangle) \otimes \mathbb{C}^*$ by twisting the $\mathbf{M}$-action by some elements $a = (a_{ij}) \in \mathrm{Sym}_{g \times g}(\mathbb{C}^*)$. Here, we have quotiented by $\langle \mathbb{B} \rangle \otimes \mathbb{C}^*$, so as to reduce redundant moduli of the general fiber as much as possible. We denote the resulting fibration by

$$f_\circ^{\mathrm{univ}}\colon X_\circ^{\mathrm{univ}}(\mathcal{S}) \to (\Delta^k)^{\mathrm{univ}}$$

where $(\Delta^k)^{\mathrm{univ}} \to (\mathrm{Sym}_{g\times g}(\mathbb{Z})/\langle \mathbb{B} \rangle) \otimes \mathbb{C}^*$ is a polydisk bundle. Note that $f_\circ^{\mathrm{univ}}\colon X_\circ^{\mathrm{univ}}(\mathcal{S}) \to (\Delta^k)^{\mathrm{univ}}$ is a locally trivial deformation of $f\colon X(\mathcal{S}) \to \Delta^k$. Indeed, as analytic germs about the zero section of $(\Delta^k)^{\mathrm{univ}}$, the universal cover of the former is the product of the universal cover of the latter with the torus of twists $(\mathrm{Sym}_{g\times g}(\mathbb{Z})/\langle\mathbb{B}\rangle)\otimes\mathbb{C}^*$. ♣

Remark 3.9. In terms of fans, the (possibly rational) origin section of $X(\mathcal{S})\to\Delta^k$ is declared to be the image of the subtorus whose cocharacter lattice is $\{0\}\times\mathbb{Z}^k\subset\mathbf{N}\times\mathbb{Z}^k$.

Remark 3.10. Let $B_1,\dots,B_k\in\mathrm{Sym}_{g\times g}(\mathbb{Z})$ and let $f^*\colon X^*(B_1,\dots,B_k)\to(\Delta^*)^k$ be the family of g-dimensional PPAVs defined in (12), with fiber X_t, $t\in(\Delta^*)^k$. Then, as we have seen above, the monodromy bilinear form about $\{u_i=0\}$ (Def. 2.6) equals B_i for a suitable symplectic basis of $H_1(X_t,\mathbb{Z})$, cf. Corollary 2.30.

Construction 3.11 (multiparameter case, cone of a fan for $\mathcal{A}_g$)**.** Suppose that the cone

$$\mathbb{B}\coloneqq\mathbb{R}_{\geq0}\{B_1,\cdots,B_k\}\subset\mathcal{P}_g^+$$

is not standard affine, or not even simplicial. Recall from Remark 2.28 that for some étale cover $\widetilde{\mathcal{A}}_g\to\mathcal{A}_g$, we have a toroidal extension $\widetilde{\mathcal{A}}_g\to\widetilde{\mathcal{A}}_g^{\mathbb{B}}$ whose monodromy cone is $\mathbb{B}$. For an appropriate choice of fan $\mathcal{S}$, Construction 3.8 gives a degeneration of abelian varieties over a k-dimensional polydisk $X(\mathcal{S})\to\Delta^k\subset Y(\mathbb{R}^k_{\geq0})$. In fact, we may descend the construction to an analytic open neighborhood of the torus fixed point in the affine toric variety $Y_{\mathbb{R}\mathbb{B}}(\mathbb{B})$, by taking a fan $\mathcal{S}$ supported rather in the vector space $\mathbf{N}_{\mathbb{R}}\times\mathbb{R}\mathbb{B}$, and periodic with respect to the same action (13). Here, the subscript $\mathbb{R}\mathbb{B}$ of $Y_{\mathbb{R}\mathbb{B}}$ refers to the fact that we take the toric variety of the polyhedral cone $\mathbb{B}\subset\mathbb{R}\mathbb{B}$, sitting inside the vector space $\mathbb{R}\mathbb{B}$, rather than inside $\mathrm{Sym}_{g\times g}(\mathbb{R})$.

Taking the universal twist, we produce a degeneration

$$X_\circ^{\mathrm{univ}}(\mathcal{S})\to T(\mathbb{B})\subset Y(\mathbb{B})$$

where now $Y(\mathbb{B})$ is the toric variety of the polyhedral cone $\mathbb{B}\subset\mathrm{Sym}_{g\times g}(\mathbb{R})$ and $T(\mathbb{B})$ is an analytic tubular neighborhood of the deepest toric boundary stratum. Then, following Section 2.5, we may analytically glue $T(\mathbb{B})$ along the complement $T^*(\mathbb{B})$ of the toric boundary to $\widetilde{\mathcal{A}}_g$. Taking the gluing to respect the zero sections, we produce a degeneration

$$X_+^{\mathrm{univ}}(\mathcal{S})\to\widetilde{\mathcal{A}}_g\cup_{T^*(\mathbb{B})}T(\mathbb{B})\eqqcolon\widetilde{\mathcal{A}}_g^+\subset\widetilde{\mathcal{A}}_g^{\mathbb{B}}$$

which analytically extends the universal family $\widetilde{\mathcal{X}}_g\to\widetilde{\mathcal{A}}_g$.

A more detailed construction of $X_+^{\mathrm{univ}}(\mathcal{S})$ would take us too far afield, but one may unify the set-up of toroidal extensions of $\mathcal{A}_g$ laid out in Section 2.5, with the toroidal construction of $X(\mathcal{S})$, by forming a fan $\mathcal{S}^{\mathrm{univ}}$ in the larger space $\mathcal{P}_g^+\oplus\mathbb{R}^g\subset\mathrm{Sym}_{g\times g}(\mathbb{R})\oplus\mathbb{R}^g$ admitting a morphism of fans $\mathcal{S}^{\mathrm{univ}}\to\mathbb{B}$ via projection to the first factor. Namikawa was the first to introduce such "mixed cone decompositions", see [49, Sec. 3] and [51, Sec. 9] (though in these texts, the focus is one particular Mumford construction, similar to Construction 6.5).

The family $X_+^{\mathrm{univ}}(\mathcal{S})$ further extends to a proper, flat family

$$f^{\mathrm{univ}}\colon X^{\mathrm{univ}}(\mathcal{S})\to\widetilde{\mathcal{A}}_g^{\mathbb{B}}\tag{16}$$

surjecting onto the base $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$. Indeed, this follows from [17, Ch. VI.1]. The idea is as follows. By taking $T(\mathbb{B}) \subset Y(\mathbb{B})$ as the maximal analytic open neighborhood of the toric boundary over which the action of $\mathbf{M}$ on the inverse image of $T(\mathbb{B})$ in $Y(\mathcal{S}^{\mathrm{univ}})$ is properly discontinuous, the base $T(\mathbb{B})$ surjects onto the toroidal extension $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$. Then, the family descends from a partial uniformization of the Baily–Borel strata of intermediate dimension, to yield (16).

Most degenerations appearing in this paper are special cases of the current construction, which are, in turn, special cases of toroidal extensions of the universal family of abelian varieties, as discussed in Faltings–Chai [17, Ch. VI.1]. The fan $\mathcal{S}$ is an instance of a "$\mathrm{GL}(X) \ltimes X^s$-admissible polyhedral cone decomposition" (for $s = 1$) in the terminology of *loc.cit.* up to the following two minor modifications: We do not demand as in [17, Ch. VI, Def.1.3.(iii)] that $\mathcal{S}$ defines a complete fan (i.e. a compactification of $\mathcal{X}_g$) and when working on an étale cover $\widetilde{\mathcal{A}}_g \to \mathcal{A}_g$ we only demand admissibility for a finite index subgroup. The constructions used to prove [17, Ch. VI.1, Thm. 1.13] generalize to this setting, to yield an extension $\widetilde{\mathcal{X}}_g^{\mathfrak{G}} \to \widetilde{\mathcal{A}}_g^{\mathfrak{F}}$ of the universal family, for any morphism $\mathfrak{G} \to \mathfrak{F}$ from a mixed cone decomposition (for $\widetilde{\mathcal{X}}_g$) to a cone decomposition (for $\widetilde{\mathcal{A}}_g$). The above family (16) is a special case.

A condition which is crucial for our applications in [27] is to achieve both a smooth total space *and* equidimensionality (i.e. flatness) over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$. As mentioned in [17, Ch. VI, Def. 1.3.(v), Rem. 1.4], achieving both of these properties is a hard combinatorial problem, one closely related to the main result of [2]; in general, it is only possible after modifying the base $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$.

As additional historical notes, the first example of a complete fan for $\mathcal{X}_g$ (equidimensional but not regular) was provided by Namikawa, see e.g. [49, Sec. 13, Prop. 13.5, Thm. 13.6]. The generalization of Faltings–Chai to the more general setting of mixed Shimura varieties is Pink's dissertation, see especially [54, Ex. 2.25, Ch. 6, Ch. 10] for discussion relevant to $\mathcal{X}_g$. ♣

As an example of the above construction, the Tate curve extends the family of elliptic curves $\mathbb{C}^*/u^{\mathbb{Z}}$ over the unit disk $\Delta_u = \{u \in \mathbb{C} : |u| < 1\}$, see Example 3.31. The maximally extended base, on which $u^{\mathbb{Z}}$ acts properly discontinuously, is $\Delta_u \supset \Delta_u^* \simeq \mathbb{Z}\backslash\mathbb{H}$. The family over Δ_u^* descends (as an orbifold) along the infinite degree surjection $\mathbb{Z}\backslash\mathbb{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, to extend the universal family over the orbifold $\mathcal{A}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Only the punctured disc $\Delta_u(e^{-2\pi})^*$ of the smaller analytic disk $\Delta_u(e^{-2\pi}) := \{u \in \mathbb{C} : |u| < e^{-2\pi}\} \subset \Delta_u$ embeds, on the level of coarse spaces, into $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (as stacks, the degree of the map from $\Delta_u(e^{-2\pi})^*$ onto its image is, rather, equal to two, because of the $\mathbb{Z}/2$-gerbe on the target).

Remark 3.12. An additional subtlety is that, to glue f^{univ} to the universal family, requires in general, a cover of $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$ which is étale in the punctured neighborhood of the boundary, but branched over the boundary divisors. For instance, when $\mathbb{B} = \mathbb{R}_{\geq 0}B$ is a ray, this branched cover is necessary if and only if B is imprimitive (Rem. 3.6). Such a branched cover of $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$ is guaranteed to exist in an affine open neighborhood of the deepest toroidal stratum, as shown in Proposition 2.29. But it is unclear, in general, whether there exists a *global* étale cover of $\widetilde{\mathcal{A}}_g$ achieving the desired branching behavior. For example, supposing the monodromy cone were of the form $\mathbb{B} = \mathbb{R}_{\geq 0}\{3B_1, 5B_2\}$ for B_1, B_2 primitive—is then the local, toroidal branched cover of $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$ which is branched to orders 3, 5 over the two toric boundary divisors $\partial_1, \partial_2 \subset \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ induced by passing to a further finite index subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$?

Regardless, this finite cover does exist in an affine open neighborhood of the relevant boundary stratum. By a slight abuse of notation, we continue to notate the resulting cover and its toroidal extension by $\widetilde{\mathcal{A}}_g$ and $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$, even though $\widetilde{\mathcal{A}}_g$ is only étale over a Zariski open subset of $\mathcal{A}_g$.

To summarize Constructions 3.8 and 3.11:

Proposition 3.13. *Let $\mathcal{S}$ be a fan in $\mathbf{N}_\mathbb{R} \times \mathbb{R}\mathbb{B}$ satisfying the properties described in Constructions 3.8 and 3.11. In particular, $\mathcal{S}$ is $\mathbf{M}$-invariant under the action (13), for a collection of symmetric bilinear forms $\{B_1, \ldots, B_k\} \subset \mathrm{Sym}^2\mathbf{M}^\vee$ which are the rays generating a polyhedral cone in the space of positive semidefinite bilinear forms on $\mathbf{M}$, with $\sum_{i=1}^k B_i$ positive-definite.*

Then Construction 3.11 produces a flat, proper extension of the universal family over $\widetilde{\mathcal{A}}_g$, in the category of complex analytic spaces,

$$X^{\mathrm{univ}}(\mathcal{S}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}.$$

We now examine when a Mumford degeneration is K-trivial:

Proposition 3.14. *If $\mathcal{S}_{(1,\ldots,1)}$ is a tiling (as opposed to a $\mathbb{Q}$-tiling) of $\mathbf{N}_\mathbb{R}$, then the multivariable Mumford construction $f\colon X(\mathcal{S}) \to \Delta^k$ (see 3.8) is K-trivial: $K_{X(\mathcal{S})} \sim 0$.*

Proof. The universal cover of $X(\mathcal{S})$ admits an analytic open embedding into the toric variety $Y(\mathcal{S})$, whose anticanonical divisor is the reduced toric boundary. This toric boundary in turn is the reduced inverse image of the union of the coordinate hyperplanes $V(u_1 \cdots u_k) \subset \mathbb{C}^k$ under the toric morphism $Y(\mathcal{S}) \to Y(\mathbb{R}^k_{\geq 0}) \simeq \mathbb{C}^k$. Thus, if the inverse image of $V(u_1 \cdots u_k)$ equals its reduced inverse image, we conclude that $Y(\mathcal{S})$ and in turn $X(\mathcal{S})$ are relatively K-trivial.

To check that the divisors contained in the inverse image of $V(u_i)$ are reduced, it suffices to restrict to the arc $\Delta \to \Delta^k$, $u \mapsto (u, \ldots, u)$. We now apply Remark 3.4. $\square$

3.2. **Weight filtration and dual complex.**

Proposition 3.15. *Let $X(\mathcal{S}) \to \Delta^k$ be a fan Mumford Construction 3.8 and let X_t be the fiber over a point $t \in (\Delta^*)^k$. Consider the exact sequence*

$$0 \to \mathbf{N} \to H_1(X_t, \mathbb{Z}) \to \mathbf{M} \to 0,$$

see (15). Then $\mathbf{N} = \mathbf{N} \subset H_1(X_t, \mathbb{Z})$ is the weight filtration $W_{-2} = W_{-1} \subset W_0$ of the limiting mixed Hodge structure on $H_1(X_t, \mathbb{Z})$. That is, there are integral isomorphisms

$$\mathbf{M} \simeq \mathrm{gr}_0^W H_1(X_t, \mathbb{Z}),$$
$$\mathbf{N} \simeq \mathrm{gr}_{-2}^W H_1(X_t, \mathbb{Z}).$$

Furthermore, $N_i \simeq N_i^{\mathrm{mon}}$ and $B_i \simeq B_i^{\mathrm{mon}}$ where $N_i^{\mathrm{mon}}\colon \mathrm{gr}_0^W H_1(X_t, \mathbb{Z}) \to \mathrm{gr}_{-2}^W H_1(X_t, \mathbb{Z})$ and $B_i^{\mathrm{mon}} \in \mathrm{Sym}^2(\mathrm{gr}_0^W H_1(X_t, \mathbb{Z}))^\vee$ are the monodromy operators and bilinear forms of Section 2.3.

Proof. The identification of $\mathbf{M}$ and $\mathbf{N}$ with the stated graded pieces of the weight filtration follows by construction, see e.g. (15)—the homology group $H_1(\mathbf{N}\otimes\mathbb{C}^*,\mathbb{Z})\simeq\mathbf{N}$ is spanned by the vanishing cycles, which are null-homologous in the neighborhood of any 0-dimensional toric stratum of $X(\mathcal{S})$. Proposition 3.7 shows that the monodromy operator N_i^{mon} agrees with $N_i\colon \mathbf{M}\to\mathbf{N}$ and the hypothesis that $B=\sum_{i=1}^k B_k>0$ ensures that $W_{-2}=(\operatorname{im}N)^{\mathrm{sat}}$ for $N=\sum_{i=1}^k N_i$ agrees with $\mathbf{N}$. □

Proposition 3.16. *Let $X(\mathcal{S})\to\Delta^k$ be a fan Mumford Construction 3.8. Then there is a canonical isomorphism $\mathbf{M}\simeq H_1(\Gamma(X_0),\mathbb{Z})$ where $\Gamma(X_0)$ is the dual (polyhedral) complex of the central fiber X_0.*

Proof. The dual polyhedral complex of the $\mathbf{M}$-prequotient is the infinite periodic polyhedral decomposition of $\mathbf{N}_\mathbb{R}$ given by the preimage of $(1,\dots,1)\in\mathbb{R}^k$ under the morphism of fans $\mathcal{S}\to(\mathbb{R}_{\geq 0})^k$, see Construction 3.8. It follows that $\Gamma(X_0)\simeq\mathcal{S}_{(1,\dots,1)}/N(\mathbf{M})=\mathbf{N}_\mathbb{R}/\Lambda_B$. Thus, there is a canonical isomorphism $H_1(\Gamma(X_0),\mathbb{Z})\simeq\mathbf{M}$. □

3.3. **Mumford construction, polytope version.** We discuss now a polytopal version of the Mumford degeneration, which outputs a relatively projective degeneration, together with a relatively ample line bundle. Furthermore, it is isomorphic to the fan construction as in Section 3.1, for an appropriate choice of fan $\mathcal{S}$ in $\mathbf{N}_\mathbb{R}\times\mathbb{R}^k$. Our approach is, in part, inspired by Gross–Siebert [31, Sec. 2], and their construction of canonical theta functions, building on the classical theory of theta functions, see e.g. [46, Prop. II.1.3 and Thm. II.1.3]. It is primarily based on a "PL version" of the classical theory, in line with Alexeev–Nakamura [7].

Let $A=\mathbb{C}^g/(\mathbb{Z}^g\oplus\mathbb{Z}^g\sigma)$ be an abelian variety with principal polarization L. The classical theory of theta functions studies explicit sections of the powers of $\mathcal{L}$, a lift of L, by pulling back to the universal cover $\pi\colon\mathbb{C}^g\to A$. Since $\pi^*\mathcal{L}\simeq\mathcal{O}_{\mathbb{C}^g}$, such sections can be understood via holomorphic functions on $\mathbb{C}^g$, with appropriate factors of automorphy under the deck action of the periods $\mathbb{Z}^g\oplus\mathbb{Z}^g\sigma$. Such holomorphic functions are called *theta functions*. We do the same here, for the intermediate cover $(\mathbb{C}^*)^g\to A$ discussed in the introduction, of the fibers $A=X_t$ of a degenerating family of PPAVs $X^*\to(\Delta^*)^k$. These theta functions extend as holomorphic sections of a line bundle over a toric extension of X^* over Δ^k.

Consider the standard torus $\mathbb{T}^g:=\mathbb{R}^g/\mathbb{Z}^g=\mathbf{M}_\mathbb{R}/\mathbf{M}$. We define $\mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L}$ to be the sheaf on $\mathbb{T}^g$ of $\mathbb{Z}$-piecewise linear functions modulo the subsheaf of $\mathbb{Z}$-linear functions. On an open set $U\subset\mathbf{M}_\mathbb{R}/\mathbf{M}$, the sections $\mathbb{Z}\mathrm{PL}(U):=\{f\colon U\to\mathbb{R}\}$ consist of continuous, piecewise linear functions, which in a domain $D\subset U$ of linearity are of the form

$$f|_D(\mathbf{m})=a_1m_1+\cdots+a_gm_g+a_{g+1} \tag{17}$$

with $a_j\in\mathbb{Z}$. Here m_i are integral coordinates on $\mathbf{M}_\mathbb{R}$ which define local coordinates on D. Similarly, $\mathbb{Z}\mathrm{L}$ is the sheaf of locally $\mathbb{Z}$-linear functions on U, of the same shape. A section $\mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L}(U)$ can be understood globally on U in terms of its "bending locus" (Definition 3.17).

The pull-back of a section $\overline{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ to the universal cover $\mathbf{M}_\mathbb{R} \to \mathbb{T}^g$ lifts to a $\mathbb{Z}$-piecewise linear function $b_i \colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$, since $H^1(\mathbf{M}_\mathbb{R}, \mathbb{Z}\mathrm{L}) = 0$. It is integer-valued on $\mathbf{M}$ and, more generally, $\frac{1}{w}\mathbb{Z}$-valued on $\frac{1}{w}\mathbf{M}$ for any positive integer $w \in \mathbb{N}$. The function $x \mapsto b_i(x+\mathbf{m}) - b_i(x)$ is a linear function on $\mathbf{M}_\mathbb{R}$ for all $\mathbf{m} \in \mathbf{M}$, because b_i is lifted from $\mathbb{T}^g$. Conversely, any $\mathbb{Z}$-piecewise linear function $b_i \colon M_\mathbb{R} \to \mathbb{R}$, such that $b_i(x+\mathbf{m}) - b_i(x)$ is linear, for all $\mathbf{m} \in \mathbf{M}$, descends to a section $\overline{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ that determines the equivalence class $[b_i]$ uniquely (where $b_i \sim b_i'$ if the difference $b_i - b_i'$ is linear).

The domains of linearity of b_i are rational polyhedra. The sections $\overline{b}_i$ for which a lifted function b_i is convex form a convex polyhedral subcone of $H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$.

Definition 3.17. Associated to $\overline{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ is a weighted polyhedral complex in $\mathbb{T}^g$, called the *bending locus* $\mathrm{Bend}(\overline{b}_i)$. Its faces are the codimension 1 polytopes in $\mathbb{T}^g$ along which $\overline{b}_i$ is non-linear, and the *bending parameter* (positive when $\overline{b}_i$ is convex) defining the weight on a codimension 1 polytope, is the change in slope of the restriction of $\overline{b}_i$ an integral, complementary segment to the hyperplane containing the face.

Definition 3.18. Let $\{\overline{b}_1, \dots, \overline{b}_k\} \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ be a collection of convex sections, with nontrivial bending in every direction, i.e. for every $\mathbf{m} \in \mathbf{M}$, there exists some $i \in \{1, \dots, k\}$ for which $b_i(x+\mathbf{m}) - b_i(x) \not\equiv 0$ is not identically zero. Equivalently, $\bigcup_i \mathrm{Bend}(\overline{b}_i)$ cuts $\mathbb{T}^g$ into polytopes. We say that the $\overline{b}_1, \dots, \overline{b}_k$ are *dicing* if the polyhedral decomposition $\bigcup_i \mathrm{Bend}(\overline{b}_i)$ has integral vertices.

The dicing condition is quite restrictive, since only the origin of $\mathbb{T}^g$ may appear as a vertex of $\bigcup_i \mathrm{Bend}(\overline{b}_i)$. We will relax this hypothesis in Construction 3.38.

Example 3.19. Let $g = 1$ and $\mathbf{M} = \mathbb{Z}$, so that $\mathbf{M}_\mathbb{R}/\mathbf{M} = \mathbb{R}/\mathbb{Z}$. Define a PL function $b \colon \mathbb{R} \to \mathbb{R}$ which is linear on each interval $[m, m+1]$, $m \in \mathbb{Z}$, and has values on $\mathbb{Z}$ equal to $b(m) = \frac{1}{2}(m^2 - m)$. The graph of b is depicted in the left of Figure 2. The locus where b is non-linear is $\mathbb{Z}$, hence $\mathrm{Bend}(\overline{b}) = \{0\} \in \mathbb{R}/\mathbb{Z} = \mathbb{T}^1$, with weight one, see Figure 4.

Remark 3.20. Any projective morphism $X \to \Delta^k$ of analytic spaces gives rise to an algebraic family $\widehat{X} \to \operatorname{Spec} \mathbb{C}[[u_1, \dots, u_k]]$, the *formal completion* of $X \to \Delta^k$. Indeed, the projectivity of $X \to \Delta^k$ implies that there is a positive integer N such that $X \subset \Delta^k \times \mathbb{P}^N$ is cut out by homogeneous polynomials whose coefficients are convergent power series. The completion $\widehat{X}$ is then cut out by the same equations, viewing the convergent power series as formal power series in $\mathbb{C}[[u_1, \dots, u_k]]$.

Construction 3.21. Let $\{\overline{b}_1, \dots, \overline{b}_k\} \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ be a collection of convex sections, with nontrivial bending in every direction, and assume that the $\overline{b}_1, \dots, \overline{b}_k$ are dicing. Let $\overline{v} \in \frac{1}{w}\mathbf{M}/\mathbf{M} \in \mathbb{T}^g$ be a $\frac{1}{w}$-integral point of $\mathbb{T}^g$, for some positive integer $w \in \mathbb{N}$. We define the *weight w theta function associated to $\overline{v}$* to be

$$\Theta_{\overline{v}}(z_1,\ldots,z_g,u_1,\ldots,u_k) \coloneqq \sum_{v\,\in\,\overline{v}+\mathbf{M}} (z_1^{x_1(v)}\cdots z_g^{x_g(v)}u_1^{b_1(v)}\cdots u_k^{b_k(v)})^w \tag{18}$$

where x_i is the i-th coordinate function, cf. [7, Sec. 4.5].

The condition that the $\overline{b}_i$ have bending in every direction ensures that this power series converges in an appropriate power series ring (which notably involves both negative and positive powers of z_i). Consider the $\mathbb{C}[[u_1,\ldots,u_k]]$-module

$$R(\overline{b}_1,\ldots,\overline{b}_k) \coloneqq \bigoplus_{w=0}^{\infty} \bigoplus_{\overline{v}\,\in\,\frac{1}{w}\mathbf{M}/\mathbf{M}} \mathbb{C}[[u_1,\ldots,u_k]]\cdot\Theta_{\overline{v}}.$$

Expanding the product of two theta functions $\Theta_{\overline{v}_1}$, $\Theta_{\overline{v}_2}$ of weights w_1, w_2 by collecting all monomial terms into $\mathbf{M}$-orbits (see (23) below), we see that there is an expansion

$$\Theta_{\overline{v}_1}\Theta_{\overline{v}_2} = \sum_{\overline{v}_3\in\frac{1}{w_1+w_2}\mathbf{M}/\mathbf{M}} c_{\overline{v}_1\overline{v}_2}^{\overline{v}_3}(u_1,\ldots,u_k)\Theta_{\overline{v}_3} \tag{19}$$

where the coefficients $c_{\overline{v}_1\overline{v}_2}^{\overline{v}_3}(u_1,\ldots,u_k)\in\mathbb{Z}[[u_1,\ldots,u_k]]$ are integral power series, as opposed to simply Laurent series, by the convexity of the b_i, see e.g. [31, Eqn. (2.5)]. Note that to get a nonzero coefficient, there must be a lift of $\overline{v}_3$ of the form

$$v_3 = \frac{w_1v_1+w_2v_2}{w_1+w_2}.$$

Hence $R(\overline{b}_1,\ldots,\overline{b}_k)$ is closed under multiplication. It is, furthermore, a finitely generated, graded ring over $\mathbb{C}[[u_1,\ldots,u_k]]$. Consider the resulting projective $\mathbb{C}[[u_1,\ldots,u_k]]$-scheme

$$\operatorname{Proj}_{\mathbb{C}[[u_1,\ldots,u_k]]} R(\overline{b}_1,\ldots,\overline{b}_k) \to \operatorname{Spec}\mathbb{C}[[u_1,\ldots,u_k]]. \tag{20}$$

It is a degeneration of PPAVs of dimension g, with the theta functions providing the projective embedding, which is the completion (in the sense of Remark 3.20) of a relatively projective complex analytic degeneration

$$f\colon X(\overline{b}_1,\ldots,\overline{b}_k)\to\Delta^k \tag{21}$$

over a polydisk. This can be proven e.g. by observing that $\Theta_{\overline{v}}$ are analytically convergent power series on a Mumford fan construction when all $|u_i|<1$, a fact which is justified in the course of the proof of Theorem 3.29. We call f the *Mumford degeneration* associated to $\{\overline{b}_1,\ldots,\overline{b}_k\}$.

Remark 3.22. That the generic fiber of (20) is an abelian variety, also follows from the classical theory of theta functions.

We have assumed that the $\overline{b}_i$ are dicing, see Definition 3.18. Define $\Gamma\subset\mathbf{M}_{\mathbb{R}}\times\mathbb{R}^k$ as the *overgraph* of the collection of functions $(b_1,\ldots,b_k)\colon\mathbf{M}_{\mathbb{R}}\to\mathbb{R}^k$, that is,

$$\Gamma = \Gamma(b_1,\ldots,b_k)+(\mathbb{R}_{\geq 0})^k\subset\mathbf{M}_{\mathbb{R}}\times\mathbb{R}^k. \tag{22}$$

Then, Γ is an infinite convex, locally finite polytope in $\mathbf{M} \times \mathbb{R}^k$, whose faces are integral polytopes. We may think of the lattice points, in $\Gamma \cap (\mathbf{M} \times \mathbb{Z}^k)$, as the monomial sections of $\mathcal{O}(1)$ on the corresponding infinite type toric variety $Y = Y_\Gamma$, see Section 2.4. Similarly, we may think of the $\frac{1}{w}$-integral points of Γ as the monomial sections of $\mathcal{O}(w)$, cf. Remark 2.21.

Then, the theta function $\Theta_{\overline{v}}$ for $\overline{v} \in \frac{1}{w}\mathbf{M}/\mathbf{M}$ is the result of summing such monomials, over an $\mathbf{M}$-orbit, where $\mathbf{m} \in \mathbf{M}$ acts on $\mathbf{M}_\mathbb{R} \times \mathbb{R}^k$ by the affine-linear action

$$(x, \vec{r}) \mapsto (x + \mathbf{m},\, \vec{r} + \vec{b}(x + \mathbf{m}) - \vec{b}(x)) \tag{23}$$

and $\vec{b} = (b_1, \ldots, b_k)$. Note that this action of $\mathbf{M}$ preserves Γ.

Form the normal fan $\mathcal{S}$ to Γ. As (23) gives an action of $\mathbf{M}$ on Γ, it induces an $\mathbf{M}$-action on $\mathcal{S}$. This action agrees with the action (13) for a multivariable Mumford fan construction, associated to the bilinear forms $B_1, \ldots, B_k \in \mathrm{Sym}^2\mathbf{M}^\vee$ defined by the equations

$$B_i(\mathbf{m}, \mathbf{m}') \coloneqq b_i(\mathbf{m} + \mathbf{m}') - b_i(\mathbf{m}) - b_i(\mathbf{m}') + b_i(0), \quad \mathbf{m}, \mathbf{m}' \in \mathbf{M}. \tag{24}$$

By the convexity of the $b_i \colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$, the normal fan $\mathcal{S} \subset (\mathbf{M}_\mathbb{R} \times \mathbb{R}^k)^\vee$ admits a canonical morphism to the fan $\mathbb{R}^k_{\geq 0} \subset \mathbb{R}^k \simeq (\mathbb{R}^k)^\vee$, given by restricting linear functionals in $(\mathbf{M}_\mathbb{R} \times \mathbb{R}^k)^\vee$ to $\mathbb{R}^k$. Then, the data of $\mathcal{S}$, together with the projection to $\mathbb{R}^k_{\geq 0}$, defines the data of a Mumford fan Construction 3.8. We will prove that $X(\overline{b}_1, \ldots, \overline{b}_k) \simeq X(\mathcal{S})$ in Theorem 3.29.

To "twist" the construction, as in Constructions 3.3, 3.8, 3.11, by some continuous parameters $a = (a_{ij}) \in \mathrm{Sym}_{g\times g}(\mathbb{C}^*)$, and produce a universal degeneration which represents all possible continuous moduli of degenerations of the specified combinatorial type, we must introduce appropriate coefficients

$$\Theta^a_{\overline{v}}(z_1, \ldots, z_g, u_1, \ldots, u_k) \coloneqq \sum_{v \,\in\, \overline{v} + \mathbf{M}} d_v(a)(z_1^{x_1(v)} \cdots z_g^{x_g(v)} u_1^{b_1(v)} \cdots u_k^{b_k(v)})^w$$

for $d_v(a) \in \mathbb{C}^*$. This twists the structure constants to give a graded ring $R^a(\overline{b}_1, \ldots, \overline{b}_k)$ and ranging over the moduli of a, produces a relatively projective multivariable Mumford degeneration over the base which is a $\operatorname{Spec} \mathbb{C}[[u_1, \ldots, u_k]]$-bundle over $(\mathbb{C}^*)^D$, $D = \dim \mathcal{A}_g - \operatorname{rank} \mathbb{R}\{B_1, \ldots, B_k\}$. It agrees on the general fiber with the quotient by the family of subgroups $au_1^{B_1} \cdots u_k^{B_k}$. These constants $d_v(a)$ form part of the so-called "degeneration data" of [17]. ♣

Notation 3.23. The construction of the ring $R(\overline{b}_1, \ldots, \overline{b}_k)$ depends only on the $\overline{b}_i$ and not the lifts b_i to PL functions on $\mathbf{M}_\mathbb{R}$. But it is usually easiest to specify $\overline{b}_i$ by providing the PL function $b_i \colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$. With this in mind, we will henceforth notate the Mumford degeneration $X(\overline{b}_1, \ldots, \overline{b}_k) \to \Delta^k$ by $X(b_1, \ldots, b_k) \to \Delta^k$, $\mathrm{Bend}(\overline{b}_i)$ by $\mathrm{Bend}(b_i)$, etc.

Remark 3.24. The number of theta functions $\Theta_{\overline{v}}$ of weight w is exactly w^g. These functions form the *theta basis*, a canonical (up to scaling) basis of sections of $H^0(X(b_1, \ldots, b_k), \mathcal{L}^{\otimes w})$ where $\mathcal{L} = \mathcal{O}(1)$ is a lift of the relative principal polarization. In particular, $\Theta = V(\Theta_{\overline{0}})$ extends as a Cartier divisor over the degenerating family $X(b_1, \ldots, b_k) \to \Delta^k$.

Since the b_i are dicing, Γ is an integer polyhedron, which is why $\mathcal{O}_Y(1)$ is Cartier on $Y = Y_\Gamma$. It also admits a natural linearization with respect to the $\mathbf{M}$-action. This is why the principal polarization extends, as a line bundle, to $X(b_1, \dots, b_k)$. Absent the dicing condition, one may consider the least positive integer d for which the overgraph Γ is a $\frac{1}{d}(\mathbf{M} \times \mathbb{Z}^k)$-integral polyhedron. Then $\mathcal{O}_Y(d)$ defines an integral polyhedron and so descends as a line bundle on $X(b_1, \dots, b_k)$ which is a lift of d times a principal polarization on the smooth fibers.

Definition 3.25. We say that $\{b_1, \dots, b_k\}$ are $\frac{1}{d}$-*dicing* if they are $\mathbb{Q}$-piecewise linear, the corresponding overgraph Γ of $\Gamma(b_1, \dots, b_k) \subset \mathbf{M}_\mathbb{R} \times \mathbb{R}^k$ is a $\frac{1}{d}(\mathbf{M} \times \mathbb{Z}^k)$-integral polyhedron, and in the local form (17), the slopes $a_1, \dots, a_g \in \mathbb{Z}$ are still integral, but we allow $a_{g+1} \in \frac{1}{d}\mathbb{Z}$. We denote sheaves of functions with such a local form by $\frac{1}{d}\mathbb{Z}\mathrm{PL}$ and $\frac{1}{d}\mathbb{Z}\mathrm{L}$.

Construction 3.26. Like Construction 3.11 vis-à-vis Construction 3.8, we generalize Construction 3.21 to the case where $\{\overline{b}_1, \dots, \overline{b}_k\}$ are the extremal rays of a convex polyhedral cone $\mathbb{b} \subset H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ mapping isomorphically to a convex polyhedral cone $\mathbb{B} \subset \mathcal{P}_g^+$, under the map $\overline{b}_i \mapsto B_i$ with $B_i \in \mathrm{Sym}^2 \mathbf{M}^\vee$ defined in (24). We replace $(b_1, \dots, b_k)$ by the PL function

$$\mathbf{M}_\mathbb{R} \to (\mathbb{R}\mathbb{b})^\vee \simeq \mathbb{R}^{\dim \mathbb{R}\mathbb{b}}$$
$$\mathbf{m} \mapsto (b \mapsto b(\mathbf{m})).$$

Otherwise, the details of Construction 3.21 are the same. The output is a relatively projective degeneration of abelian varieties

$$X(\mathbb{b}) \to T(\mathbb{b}) \subset Y(\mathbb{b})$$

over an analytic tubular neighborhood of the torus fixed point of $Y(\mathbb{b})$. Performing this construction with the universal twist by $a \in \mathrm{Sym}_{g\times g}(\mathbb{Z})/\langle \mathbb{B} \rangle \otimes \mathbb{C}^*$, for $\mathbb{B} = \mathbb{R}_{\geq 0}\{B_1, \dots, B_k\}$, and extending/descending over the toroidal extension $\widetilde{\mathcal{A}}_g^\mathbb{B}$ as in Construction 3.11, we may produce a relatively projective analytic extension of the universal family

$$X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^\mathbb{B},$$

see Proposition 3.13. We will show in Section 6 that $X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^\mathbb{B}$ is an étale-locally projective morphism of algebraic spaces over $\widetilde{\mathcal{A}}_g^\mathbb{B}$. ♣

Remark 3.27. Our primary case of interest in [27] is where $\mathbb{b}$ defines a simplicial cone $\mathbb{B} \subset \mathcal{P}_g^+$ and in this setting, Constructions 3.8 and 3.21 over a polydisk will suffice.

The next proposition follows directly from Construction 3.21:

Proposition 3.28. *Let $(\mathbb{T}^g, b_1, \dots, b_k)$ define a polarized, multivariable Mumford degeneration $X(b_1, \dots, b_k) \to \Delta^k$. Then the following hold:*

(1) *The intersection complex of the fiber X_I over the generic point of the coordinate subspace $V(u_i : i \in I)$ is the polyhedral decomposition $\bigcup_{i\in I} \mathrm{Bend}(\overline{b}_i)$ of $\mathbb{T}^g$.*

(2) *The polytopes of this polyhedral decomposition, when compact, are the polytopes of the polarized toric components, in the sense of Remark 2.21.*
(3) *Non-compact faces F of $\bigcup_{i\in I}\mathrm{Bend}(\bar{b}_i)$ are of the form $F\simeq F_0\times\mathbb{T}^h$ with F_0 compact. The dimension of the abelian part of the corresponding component of X_I is h. This component is a toric variety bundle over an abelian h-fold, possibly self-glued, where the toric variety has polytope F_0.*

Sketch. The universal cover of the Mumford construction is the toric variety Y_Γ whose polytope is Γ, and hence Γ is the intersection complex of this universal cover. Then the intersection complex of the Mumford construction itself is the quotient by the $\mathbf{M}$ action, and the stated description follows—components with an abelian factor of dimension $h>0$ arise from infinite faces of Γ stabilized by a rank h subgroup of $\mathbf{M}$. See also Theorem 3.29. □

3.4. **Comparison of polytope and fan constructions.** We explain why the polytope construction of the Mumford degeneration coincides with the fan construction.

Theorem 3.29. *Let $\{\bar{b}_1,\dots,\bar{b}_k\}\in H^0(\mathbb{T}^g,\mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ be a collection of convex sections, with nontrivial bending in every direction, which are dicing. Define $\Gamma\subset\mathbf{M}_\mathbb{R}\times\mathbb{R}^k$ as in (22), and let $\mathcal{S}$ be the normal fan to Γ. Then, there is a canonical isomorphism of analytic spaces*

$$X(\bar{b}_1,\dots,\bar{b}_k)\simeq X(\mathcal{S})$$

over Δ^k, where $X(\bar{b}_1,\dots,\bar{b}_k)\to\Delta^k$ is the polytope Mumford degeneration defined in (21) and where $X(\mathcal{S})\to\Delta^k$ is the fan Mumford degeneration defined in (14).

Proof. Let $Y=Y_\Gamma$ be the locally finite type toric variety defined by the polytope Γ, as in Section 2.4. Then $Y=Y(\mathcal{S})$ by definition, and the $\mathbf{M}$-action (23) on Γ, which we denote $\mathbf{m}\cdot -$, defines a linearization of the line bundle $\mathcal{O}_Y(1)$ associated to the polytope Γ. A $\frac{1}{w}$-integral point $(v,\vec{r})\in\Gamma\cap\frac{1}{w}(\mathbf{M}\times\mathbb{Z}^k)$ defines an analytic section $(z,u)^{w(v,\vec{r})}:=z^{wv}u^{w\vec{r}}\in H^0(Y,\mathcal{O}_Y(w))$. When this $\frac{1}{w}$-integral point lies on the graph $\Gamma(b_1,\dots,b_k)$, we have the equality

$$\Theta_{\overline{v}}(z,u)=\sum_{\mathbf{m}\in\mathbf{M}}(z,u)^{w(\mathbf{m}\cdot(v,\vec{r}))} \tag{25}$$

of analytic functions, on the analytic open subset of Y where all $|u_i|<1$. Convergence holds because the b_i having nontrivial bending in all directions (see Definition 3.18), and so with respect to an exhaustion of $\mathbf{M}$, the powers of u grow quadratically while the powers of z only grow linearly. If, rather, $(v,\vec{r})$ lies above the graph $\Gamma(b_1,\dots,b_k)$, the corresponding sum over the $\mathbf{M}$-orbit is simply a monomial in u times $\Theta_{\overline{v}}(z,u)$. We deduce that $\Theta_{\overline{v}}(z,u)$ descends, as an analytic section, to $H^0(X(\mathcal{S}),\mathcal{L}^{\otimes w})$ where $\mathcal{L}$ is the descent of the $\mathbf{M}$-linearized line bundle $\mathcal{O}_Y(1)$ to the $\mathbf{M}$-quotient $X(\mathcal{S})$.

It suffices then to verify that these descended sections define, for some fixed w, a relatively very ample line bundle on $X(\mathcal{S})$—in particular, that they separate points and tangents. The argument is essentially the same as [7, Thm. 4.7], replacing the Delaunay decomposition with the more general decompositions $\bigcup_i \mathrm{Bend}(\overline{b}_i)$ that we consider.

In fact, in the setting where all polytopal faces F cut by $\bigcup_i \mathrm{Bend}(\overline{b}_i)$ are embedded as opposed to immersed in $\mathbb{T}^g$, the multiplication rule (19) for theta functions $\Theta_{\overline{v}}$ for $\overline{v} \in F$, reduce, modulo the ideal $(u_1, \dots, u_k)$, to the usual multiplication rule (Rem. 2.21) for the monomial sections of the powers of the line bundle $\mathcal{L}|_{Y_F}$ on the toric stratum $Y_F \subset X(\mathcal{S})$ associated to F; see Lemma 3.33. Assuming w is sufficiently large, we also ensure that the non-normal union $X_0(\mathcal{S}) = \lim_F Y_F$ is projectively embedded via $\mathcal{L}^{\otimes w}$. One deduces very ampleness for all fibers, by the openness of very ampleness. □

For instance, we have the following special case, for 1-parameter degenerations:

Corollary 3.30. *Let $\overline{b} \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ be dicing, with PL lift $b\colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$ and define $\Gamma := \Gamma(b) + \mathbb{R}_{\geq 0} \subset \mathbf{M}_\mathbb{R} \times \mathbb{R}$. Let $\mathcal{S}$ be the normal fan to Γ, and define $B \in \mathrm{Sym}^2\mathbf{M}^\vee$ by $B(\mathbf{m}, \mathbf{m}') := b(\mathbf{m}+\mathbf{m}') - b(\mathbf{m}) - b(\mathbf{m}') + b(0)$. Define a Λ_B-invariant tiling $\mathcal{T}$ of $\mathbf{N}_\mathbb{R}$ by slicing $\mathcal{S}$ at height 1. Then $X(\mathrm{Cone}(\mathcal{T})) \simeq X(\mathcal{S}) \simeq X(\overline{b})$.* □

3.5. **Examples.** We now discuss some examples of the Mumford construction.

Example 3.31. The basic example is the Tate curve $\mathbb{C}^*/u^\mathbb{Z}$. Here $g = 1$, so $\mathbf{M}_\mathbb{R}/\mathbf{M} \simeq \mathbb{R}/\mathbb{Z}$. Define a PL function $b\colon \mathbb{R} \to \mathbb{R}$ which is linear on each interval $[m, m+1]$, $m \in \mathbb{Z}$, and has values on $\mathbb{Z}$ equal to $b(m) = \frac{1}{2}(m^2 - m)$. The graph of b is depicted in the left of Figure 2.

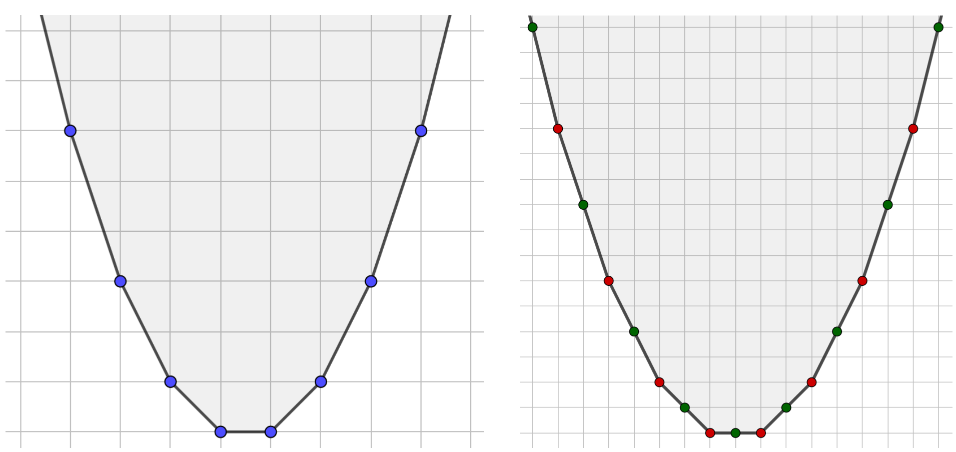


FIGURE 2. Mumford polytope construction of the Tate curve, $\Theta_{0/1}(z,u)$ in blue, $\Theta_{0/2}(z,u)$ in red, $\Theta_{1/2}(z,u)$ in green.

Then $\Gamma = \{(m, \frac{1}{2}(m^2 - m)) : m \in \mathbb{Z} = \mathbf{M}\} \subset \mathbf{M} \times \mathbb{Z} = \mathbb{Z}^2$ and Γ is the shaded region in Figure 2. The theta functions of weight 1, 2, and 3 are

$$\begin{aligned}
\Theta_{0/1}(z,u) &= \cdots + z^{-3}u^6 + z^{-2}u^3 + z^{-1}u^1 + z^0u^0 + z^1u^0 + z^2u^1 + z^3u^3 + \cdots \\
\Theta_{0/2}(z,u) &= \cdots + z^{-6}u^{12} + z^{-4}u^6 + z^{-2}u^2 + z^0u^0 + z^2u^0 + z^4u^2 + z^6u^6 + \cdots \\
\Theta_{1/2}(z,u) &= \cdots + z^{-5}u^9 + z^{-3}u^4 + z^{-1}u^1 + z^1u^0 + z^3u^1 + z^5u^4 + z^7u^9 + \cdots \\
\Theta_{0/3}(z,u) &= \cdots + z^{-9}u^{18} + z^{-6}u^9 + z^{-3}u^3 + z^0u^0 + z^3u^0 + z^6u^3 + z^9u^9 + \cdots \\
\Theta_{1/3}(z,u) &= \cdots + z^{-8}u^{15} + z^{-5}u^7 + z^{-2}u^2 + z^1u^0 + z^4u^1 + z^7u^5 + z^{10}u^{12} + \cdots \\
\Theta_{2/3}(z,u) &= \cdots + z^{-7}u^{12} + z^{-4}u^5 + z^{-1}u^1 + z^2u^0 + z^5u^2 + z^8u^7 + z^{11}u^{15} + \cdots
\end{aligned}$$

with those of weight 1 and 2 depicted in Figure 2 as the sum of the blue, red, and green monomials. The normal fan is depicted in Figure 3 and the tiling $\mathcal{T}$ (i.e. the slice of the normal fan at height 1) is the tiling of $\mathbb{R}^1/\Lambda_B$ by a segment of length 1. Here $\Lambda_B \simeq \mathbb{Z}$ because the bilinear form on $\mathbb{R}^1$ defined by the formula

$$B(x,y) = \tfrac{1}{2}(x+y)^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 = xy$$

has 1×1 Gram matrix $[1] \in \mathrm{Sym}_{1\times 1}(\mathbb{Z})$.

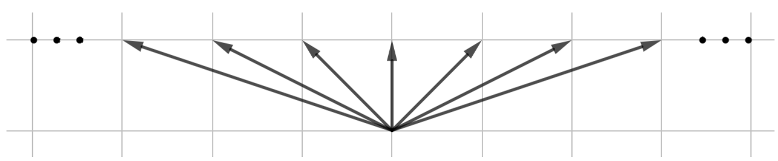


FIGURE 3. Normal fan of the Tate curve.

The torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and the weighted polyhedral complex $\mathrm{Bend}(b)$ inside it are depicted in Figure 4 (the most condensed presentation of a Mumford construction). This figure happens to be the same as the tiling $\mathcal{T}$, but this is a coincidence. By Proposition 3.28, this polyhedral decomposition of $\mathbb{T}^1$ is the intersection complex of the special fiber X_0 of the degeneration of elliptic curves $X = X(b) \to \Delta$, with strata formed from the polytopes of the decomposition $\mathrm{Bend}(b)$. Hence X_0 is $\mathbb{P}^1$ glued to itself along two points, 0 and ∞.

Figure 5 is a visual depiction of Construction 3.3. By considering the maximal cones of the normal fan of Figure 4, we see that the universal cover $Y(\mathrm{Cone}(\mathcal{T})) \to \Delta_u$ of the Tate curve may be constructed as an infinite union of copies of $\mathbb{C}^2$:

$$Y(\mathrm{Cone}(\mathcal{T})) = \bigcup_{n\in\mathbb{Z}} \mathbb{C}^2_{(x_n,y_n)}$$

where the gluings are $x_{n+1} = y_n^{-1}$ and $y_{n+1} = x_n y_n^2$. The map to $\mathbb{C}_u$ is given on local charts by $u = x_n y_n$ and respects the gluings. Finally, the $\mathbb{Z}$-action is $(x_n, y_n) \mapsto (x_{n+1}, y_{n+1})$.

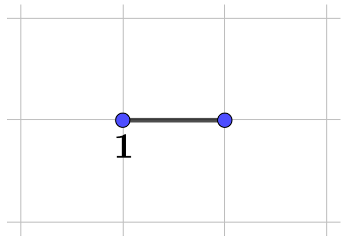


FIGURE 4. Bending complex of the Tate curve in $\mathbb{R}/\mathbb{Z}$. The integer 1 indicates the bending parameter, see Definition 3.17.

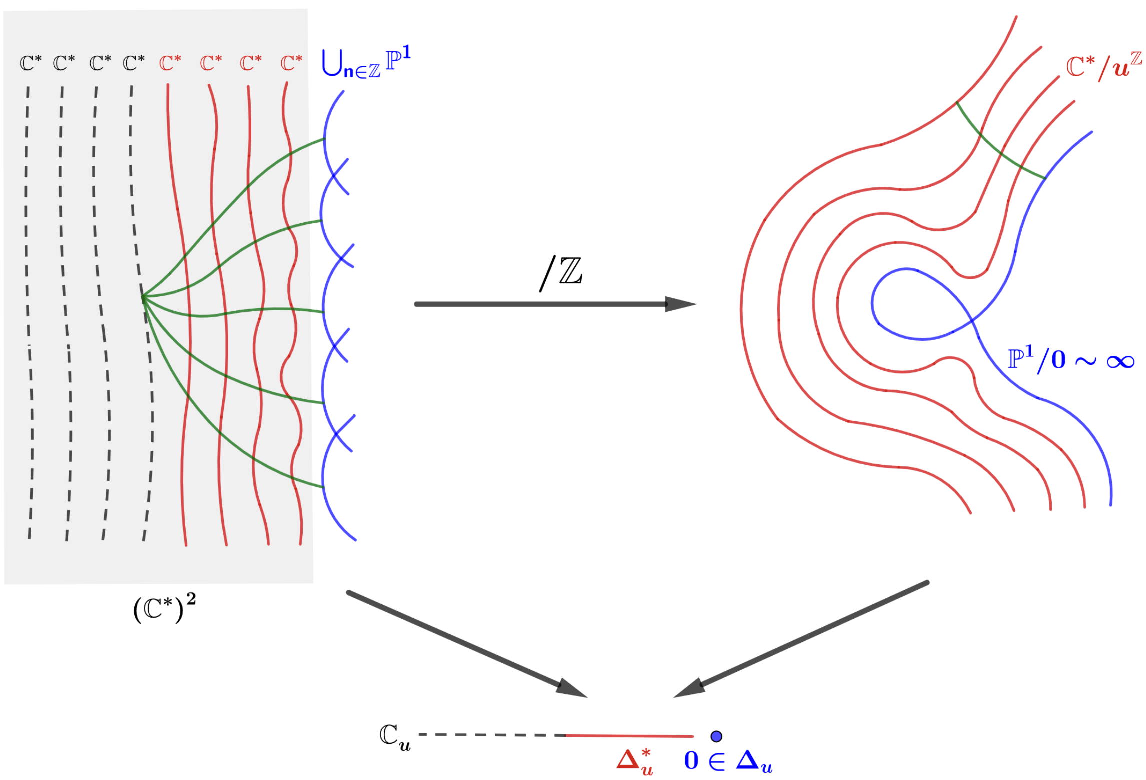


FIGURE 5. Top left: Universal cover of the Tate curve. Inverse image of Δ_u^* depicted in red, with embedding into $(\mathbb{C}^*)^2$, in grey. In the toroidal extension $Y(\mathrm{Cone}(\mathcal{T}))$, the fiber over $0 \in \Delta_u$ in blue is an infinite $\mathbb{Z}$-periodic quilt of toric varieties, given by gluing an infinite chain of $\mathbb{P}^1$s. A $\mathbb{Z}$-orbit of co-characters passing through $(1,1) \in (\mathbb{C}^*)^2$ and forming sections over $\mathbb{C}_u$ is depicted in green. Top right: The Tate curve, with general fiber $\mathbb{C}^*/u^{\mathbb{Z}}$ in red, central nodal fiber in blue, and section in green.

Example 3.32 (Multiplication of theta functions)**.** In general the multiplication rule for theta functions is quite complicated, but here we check that for the central fiber $u = 0$ of the Tate curve, Example 3.31, the sections of $\mathcal{L}^{\otimes 3}$ define an embedding of a nodal cubic $X_0(b) \hookrightarrow \mathbb{P}^2$. In addition, the computation will show that the generic fiber of $\mathrm{Proj}_{\mathbb{C}[[u]]} R(b) \to \operatorname{Spec} \mathbb{C}[[u]]$, see (20), is a smooth cubic plane curve, as also follows from Theorem 3.29. See Remark 3.22.

By quotienting by the ideal (u), the multiplication rule (19) significantly simplifies. Generally, a product of two monomials lies in the ideal $(u_1, \dots, u_k)$, whenever they lie over distinct domains of linearity of the b_i as then their product, viewed as a lattice point in $\mathbf{M} \times \mathbb{Z}^k$, lies strictly above the graph $\Gamma(b_1, \dots, b_k)$ by convexity. We deduce:

Lemma 3.33. *$\Theta_{\overline{v}_1} \cdot \Theta_{\overline{v}_2} = 0$ mod $(u_1, \dots, u_k)$ whenever $\overline{v}_1$, $\overline{v}_2$ do not lie in any common polyhedral domain of $\bigcup_i \mathrm{Bend}(\overline{b}_i) \subset \mathbb{T}^g$. Furthermore, if both $\overline{v}_1$, $\overline{v}_2$ lie in the interior of a polyhedral domain of maximal dimension, then we have*

$$\Theta_{\overline{v}_1} \cdot \Theta_{\overline{v}_2} = \Theta_{\frac{w_1 \overline{v}_1 + w_2 \overline{v}_2}{w_1 + w_2}} \bmod (u_1, \dots, u_k).$$

Otherwise, the multiplication rule mod $(u_1, \dots, u_k)$ must take into account the fact that there are might be multiple representatives v_1, v_2 of $\overline{v}_1, \overline{v}_2$ which lie in the same domain of linearity. In any case, we may apply this comment and Lemma 3.33 to the weight 3 theta functions of Example 3.31. We deduce the following multiplication rules mod u:

$$\begin{aligned}
\Theta_{0/3}^3 &= \Theta_{0/9} + 3\Theta_{3/9} + 3\Theta_{6/9} & \Theta_{1/3}^3 &= \Theta_{3/9} & & \\
\Theta_{0/3}^2\Theta_{1/3} &= \Theta_{1/9} + 2\Theta_{4/9} + \Theta_{7/9} & \Theta_{1/3}^2\Theta_{2/3} &= \Theta_{4/9} & \Theta_{0/3}\Theta_{1/3}^2 &= \Theta_{2/9} + \Theta_{5/9} \\
\Theta_{0/3}^2\Theta_{2/3} &= \Theta_{2/9} + 2\Theta_{5/9} + \Theta_{8/9} & \Theta_{1/3}\Theta_{2/3}^2 &= \Theta_{5/9} & \Theta_{0/3}\Theta_{2/3}^2 &= \Theta_{4/9} + \Theta_{7/9} \\
\Theta_{0/3}\Theta_{1/3}\Theta_{2/3} &= \Theta_{3/9} + \Theta_{6/9} & \Theta_{2/3}^3 &= \Theta_{6/9} & &
\end{aligned}$$

as only $0/3 \in \frac{1}{3}\mathbf{M}/\mathbf{M} \subset \mathbb{R}/\mathbb{Z}$ lies in multiple domains of linearity of $\overline{b}$.

These are the ten cubics in $\mathrm{Sym}^3 H^0(X_0(b), \mathcal{L}^{\otimes 3})$, and there is indeed one linear relation:

$$\Theta_{0/3}\Theta_{1/3}\Theta_{2/3} = \Theta_{1/3}^3 + \Theta_{2/3}^3 \bmod u,$$

which gives the projective equation $\{xyz = x^3 + y^3\} \subset \mathbb{P}^2$ of a cubic curve, with a simple node at $[0:0:1]$. Computing the expansion of theta products rather over $\mathbb{C}[[u]]/(u^2)$, one additionally sees that the local equation of the node in the total space is $xy = u$ and hence the general fiber of $X(b) \to \Delta$ is a smooth elliptic curve.

Example 3.34 (Theta graph, 1-parameter)**.** Now consider the 1-parameter Mumford degeneration $(\mathbb{T}^2, b)$ corresponding to the PL function

$$b(m_1, m_2) = r_1 \frac{m_1^2 - m_1}{2} + r_2 \frac{m_2^2 - m_2}{2} + r_3 \frac{(m_1 + m_2)^2 - (m_1 + m_2)}{2}$$

on $\mathbf{M} = \mathbb{Z}^2$ where $(r_1, r_2, r_3) \in \mathbb{N}^3$ is some fixed vector. Then b is convex and the boundary of Γ is depicted in Figure 6.

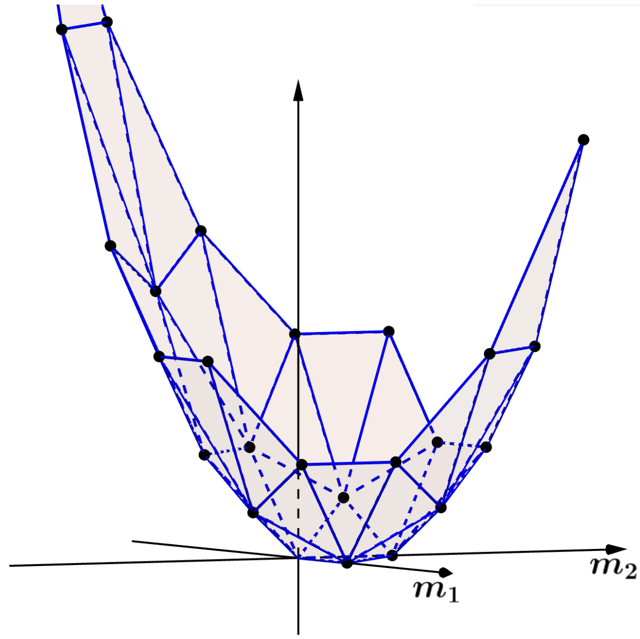


FIGURE 6. The graph of b over $\mathbf{M}_{\mathbb{R}} = \mathbb{R}^2$, for values $r_1 = r_2 = r_3 = 1$.

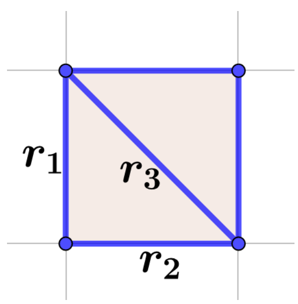


FIGURE 7. Bending complex of b in $\mathbb{T}^2$.

We have, for instance, the weight 1 theta function

$$\begin{aligned}\Theta_{(0/1,0/1)}(z_1,z_2,u) = \cdots &+ z_1^{-1}z_2^{1}\;\; u^{r_1} && + z_1^0 z_2^1\;\; u^0 && + z_1^1 z_2^1\;\; u^{r_3}\\ &+ z_1^{-1}z_2^{0}\;\; u^{r_1+r_3} && + z_1^0 z_2^0\;\; u^0 && + z_1^1 z_2^0\;\; u^0\\ &+ z_1^{-1}z_2^{-1}u^{r_1+r_2+3r_3} && + z_1^0 z_2^{-1}u^{r_2+r_3} && + z_1^1 z_2^{-1}u^{r_2} + \cdots\end{aligned}$$

e.g. since $b(-1,-1) = r_1 + r_2 + 3r_3$. The bending complex of b is depicted in Figure 7. The associated bilinear form has Gram matrix

$$B = r_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + r_2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + r_3\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

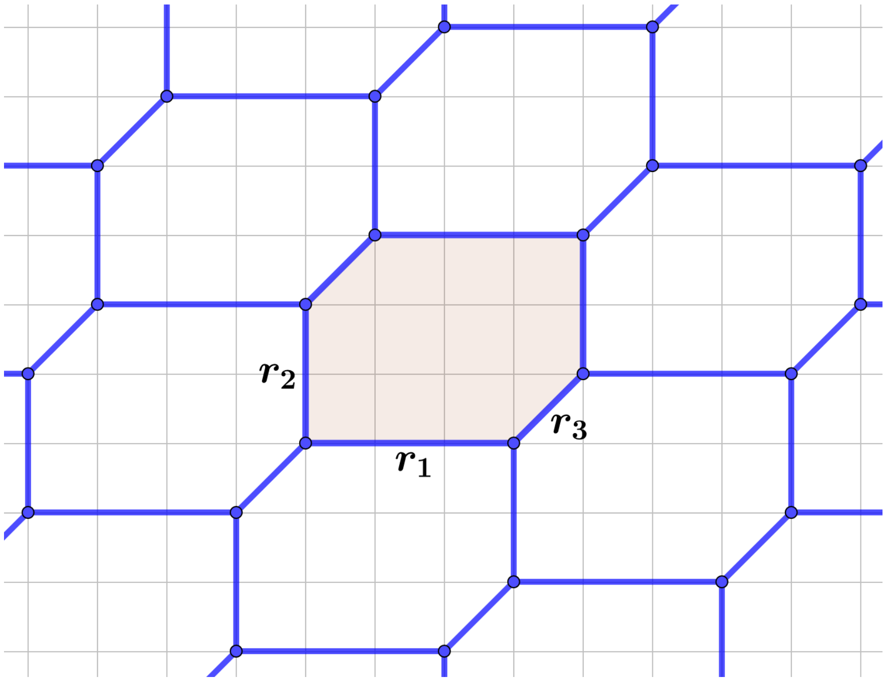


FIGURE 8. Fundamental domain for the Λ_B-action on $\mathbb{R}^2$, $(r_1, r_2, r_3) = (3, 2, 1)$, and tiling $\mathcal{T}$, arising from slicing the normal fan at height 1.

and the tiling $\mathcal{T}$ is depicted in Figure 8.

To produce the universal 1-parameter Mumford degeneration, requires twisting the construction by $a \in \mathrm{Sym}_{2\times 2}(\mathbb{C}^*)/u^B \simeq (\mathbb{C}^*)^2$.

Example 3.35 (Theta graph, 3-parameter)**.** We now modify the previous example, by instead taking a 3-parameter Mumford degeneration for $(\mathbb{T}^2, b_1, b_2, b_3)$ where

$$b_1(m_1, m_2) := \frac{m_1^2 - m_1}{2}, \quad b_2(m_1, m_2) := \frac{m_2^2 - m_2}{2}, \quad b_3(m_1, m_2) := \frac{(m_1+m_2)^2 - (m_1+m_2)}{2}.$$

This example is originally attributed to Deligne [44, Sec. 7]; called by Mumford the "keystone" of the compactification of $\mathcal{A}_2$. The figure is similar to Figure 6, but we now use three different colors, to indicate the different bending loci $\mathrm{Bend}(b_i)$, for $i = 1, 2, 3$. See Figure 9.

The theta function of weight 1 is

$$\begin{aligned}\Theta_{(0/1,0/1)}(z_1, z_2, u_1, u_2, u_3) = \cdots &+ z_1^{-1} z_2^1 u_1^1 u_2^0 u_3^0 &&+ z_1^0 z_2^1 u_1^0 u_2^0 u_3^0 &&+ z_1^1 z_2^1 u_1^0 u_2^0 u_3^1 \\ &+ z_1^{-1} z_2^0 u_1^1 u_2^0 u_3^1 &&+ z_1^0 z_2^0 u_1^0 u_2^0 u_3^0 &&+ z_1^1 z_2^0 u_1^0 u_2^0 u_3^0 \\ &+ z_1^{-1} z_2^{-1} u_1^1 u_2^1 u_3^3 &&+ z_1^0 z_2^{-1} u_1^0 u_2^1 u_3^1 &&+ z_1^1 z_2^{-1} u_1^0 u_2^1 u_3^0 + \cdots .\end{aligned}$$

Note that, upon restriction to the co-character $\mathrm{Spec}\,\mathbb{C}[[u]] \subset \mathrm{Spec}\,\mathbb{C}[[u_1, u_2, u_3]]$ defined by $(u_1, u_2, u_3) = (u^{r_1}, u^{r_2}, u^{r_3})$ with $(r_1, r_2, r_3) \in \mathbb{N}^3$, we get $\Theta_{(0/1,0/1)}(z_1, z_2, u)$ from Example

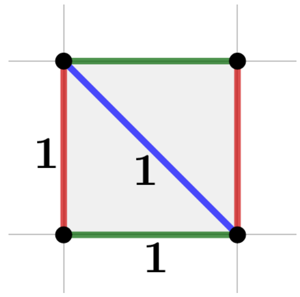


FIGURE 9. Bending complexes of b_1, b_2, b_3 in $\mathbb{T}^2$, in red, green, and blue, respectively. The integers are the bending parameters of b_1, b_2, and b_3.

3.34, and indeed, the multivariable Mumford construction restricts to a 1-parameter one along this co-character, and $B = r_1B_1 + r_2B_2 + r_3B_3$.

To understand the fibers of $X(b_1, b_2, b_3) = X \to \Delta^3$ over the various coordinate subspaces, we refer to Figure 10. By Proposition 3.28, the polytopes of the components of the fiber over $(u_1, u_2, u_3) \in \Delta^3$ of the Mumford construction can be read off from the bending loci of the b_i for which $V(u_i) = 0$. The fiber over the origin is the union

$$X_{(0,0,0)} = \mathbb{P}^2 \cup_\triangle \mathbb{P}^2$$

of two copies of $\mathbb{P}^2$ along a triangle of lines, so that the intersection complex is the upper left of Figure 10. The limit of the theta divisor is the union of two lines

$$V(\Theta_{(0/1,\,0/1)}) \cap X_{(0,0,0)} = \ell_1 \cup \ell_2 \subset \mathbb{P}^2 \cup_\triangle \mathbb{P}^2.$$

The fiber $X_{(0,0,u_3)}$ over a point on the u_3-axis is normalized by the square $(\mathbb{P}^1 \times \mathbb{P}^1, \square)$ and results from gluing two sections (the top and bottom of the square) and two fibers (the left and right of the square). The gluing isomorphisms are $u_3 \in \mathbb{C}^*$, $u_3^{-1} \in \mathbb{C}^*$. The theta divisor $V(\Theta_{(0/1,\,0/1)})$ lies in the linear system of $\mathcal{O}(1,1)$ on the normalization, and glues to a Cartier divisor on the non-normal surface $X_{(0,0,u_3)}$. The fibers over the u_1- and u_2-axes of Δ^3 are similar; the intersection complexes are in the top row of Figure 10.

Over a general point of a coordinate hyperplane $V(u_1)$, the fiber $X_{(0,u_2,u_3)}$ is the result of gluing a $\mathbb{P}^1$-bundle $\mathbb{P}_E(\mathcal{O} \oplus \mathcal{M})$ over an elliptic curve E to itself, $\mathcal{M} \in \mathrm{Pic}^0(E)$, by gluing the 0- and ∞-sections of the bundle by a translation depending on $\mathcal{M}$. Fibers over the other coordinate hyperplanes are similar, and the corresponding intersection complexes are the first three figures in the second row of Figure 10. Finally, the general fiber $X_{(u_1,u_2,u_3)}$ over a point $(u_1, u_2, u_3) \in (\Delta^*)^3$ is a smooth, principally polarized abelian surface with $V(\Theta_{(0/1,\,0/1)})$ the theta divisor.

3.6. **Base change and Veronese embedding.** Let $X(b_1, \ldots, b_k) \to \Delta^k$ be a Mumford construction on a collection of convex sections $b_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$, see Construction 3.21, and

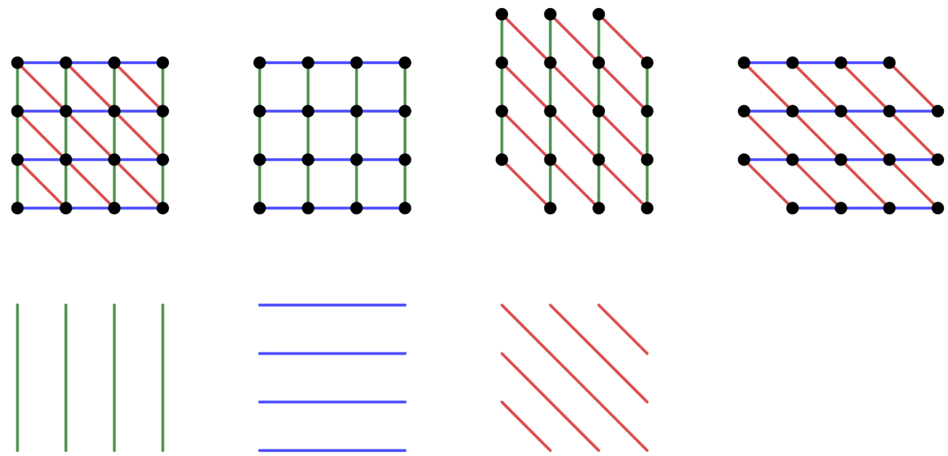


FIGURE 10. Intersection complexes of the fibers of the Mumford construction.

consider a monomial base change, of the form $\Delta^n \to \Delta^k$, such that the pullback of coordinates u_i are of the form

$$\begin{aligned} u_1 &= w_1^{r_{11}} \cdots w_n^{r_{1n}} =: w^{\vec{r}_1}, \\ &\cdots \\ u_k &= w_1^{r_{k1}} \cdots w_n^{r_{kn}} =: w^{\vec{r}_k}. \end{aligned} \tag{26}$$

Proposition 3.36. *The base change of the Mumford degeneration $X(b_1, \dots, b_k) \to \Delta^k$ along a monomial base change $\Delta^n \to \Delta^k$ is the Mumford construction $X(c_1, \dots, c_n) \to \Delta^n$ associated to the convex PL functions*

$$c_j := r_{1j} b_1 + \cdots + r_{kj} b_k.$$

Proof. Substituting (26) into the defining equations for $\Theta_{\overline{v}}$ in (18), we see that the result is again a Mumford degeneration $X(c_1, \dots, c_n) \to \Delta^n$ where c_j has the stated formula. $\square$

A simple case is exhibited by Examples 3.34, 3.35, where we make the monomial base change $\Delta \to \Delta^3$, $w \mapsto (w^{r_1}, w^{r_2}, w^{r_3})$ to Example 3.35, to get Example 3.34.

Remark 3.37. A base change $\Delta^k \to \Delta^k$ ramified over the coordinate hyperplane $V(u_i)$ to order r_i is given by $u_i = w_i^{r_i}$ and we have the simpler relation $c_i = r_i b_i$.

We now consider the effect of replacing L with dL for some positive multiple $d > 0$ of the principal polarization. Equivalently, we are taking the Veronese subring

$$R(b_1, \dots, b_k)^{(d)} \subset R(b_1, \dots, b_k)$$

consisting of the theta functions of weights w divisible by d. The passage to the Veronese subring suggests also a natural generalization of Construction 3.21, which allows us to relax the restrictive dicing condition:

Construction 3.38. Consider convex sections

$$\bar{b}_i \in H^0(\mathbb{T}^g, \tfrac{1}{d}\mathbb{Z}\mathrm{PL}/\tfrac{1}{d}\mathbb{Z}\mathrm{L})$$

for some positive integer d, see Definition 3.18. Then, we may define theta functions similarly to formula (18), but only for the weights w divisible by d. Assuming that the $\bar{b}_i$ are $\frac{1}{d}$-dicing on $\mathbb{T}^g$ (Def. 3.25), we may deduce from Remark 3.24 that dL lifts to an ample line bundle on the resulting degeneration, which we denote by

$$X(d \mid b_1, \dots, b_k) \to \Delta^k.$$

While the general fiber of the degeneration still admits a principal polarization L, only dL extends to a line bundle on the total space, in general.

In terms of the polytope, we still take the overgraph $\Gamma = \Gamma(b_1, \dots, b_k) + (\mathbb{R}_{\geq 0})^k$, which is now only a $\frac{1}{d}(\mathbf{M} \times \mathbb{Z}^k)$-integral polyhedron. Then, as in Remark 2.21, we consider $\mathrm{Cone}(\Gamma) \subset (\mathbf{M}_{\mathbb{R}} \times \mathbb{R}^k) \times \mathbb{R}$ but one only considers monomials, and their $\mathbf{M}$-averagings (25) to theta functions, lying in $(\mathbf{M} \times \mathbb{Z}^k) \times d\mathbb{Z}$. The general fiber of the Mumford construction is still the quotient by $\mathbf{M}$ of $\mathbf{N} \otimes \mathbb{C}^*$; in particular, the exact sequence (10) still holds. ♣

Remark 3.39. The isomorphism type of the degeneration $X(d \mid b_1, \dots, b_k) \to \Delta^k$ does not depend on the lifts b_i of $\bar{b}_i$ but the choice of origin section does, since different lifts of $\bar{b}_i$ shift the normal fan, and thus affect which subtorus forms the origin section, see Remark 3.9. The same applies to Construction 3.26—to produce an extension of the universal family over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$ requires a choice of lift of cone $\mathbb{b} \subset H^0(\mathbb{T}^g, \frac{1}{d}\mathbb{Z}\mathrm{PL}/\frac{1}{d}\mathbb{Z}\mathrm{L})$ into $H^0(\mathbf{M}_{\mathbb{R}}, \frac{1}{d}\mathbb{Z}\mathrm{PL})$ as the gluing with the universal family $\widetilde{\mathcal{X}}_g \to \widetilde{\mathcal{A}}_g$ depends on a choice of origin section.

Example 3.40 (Base change and resolution of the Tate curve)**.** Consider the following Mumford constructions of degenerating elliptic curves:

(1) The Tate curve, i.e. Example 3.31.
(2) The order 3 base change $u = w^3$ to Example 3.31.
(3) The order 3 Veronese embedding of (2).

These are encoded respectively by the following data:

(1) $\mathbb{R}/\mathbb{Z}$, $b^{(1)} = \frac{1}{2}(m^2 - m)$ on $m \in \mathbb{Z}$, $d = 1$.
(2) $\mathbb{R}/\mathbb{Z}$, $b^{(2)} = \frac{3}{2}(m^2 - m)$ on $m \in \mathbb{Z}$, $d = 1$.
(3) $\mathbb{R}/\mathbb{Z}$, $b^{(3)} = b^{(2)}$, $d = 3$ (see the outer edge of Figure 11).

The total space of the Tate curve $X(b^{(1)})$ is smooth and the central fiber $X_0(b^{(1)})$ is an irreducible nodal curve (i.e. of Kodaira type I_1). The total space of the base change $X(b^{(2)})$ is singular, with an A_2-singularity at the node point of the irreducible central fiber. The spaces $X(3 \mid b^{(2)}) \simeq X(b^{(2)})$ are isomorphic degenerations over Δ, with the former polarized by $3L$ rather than L.

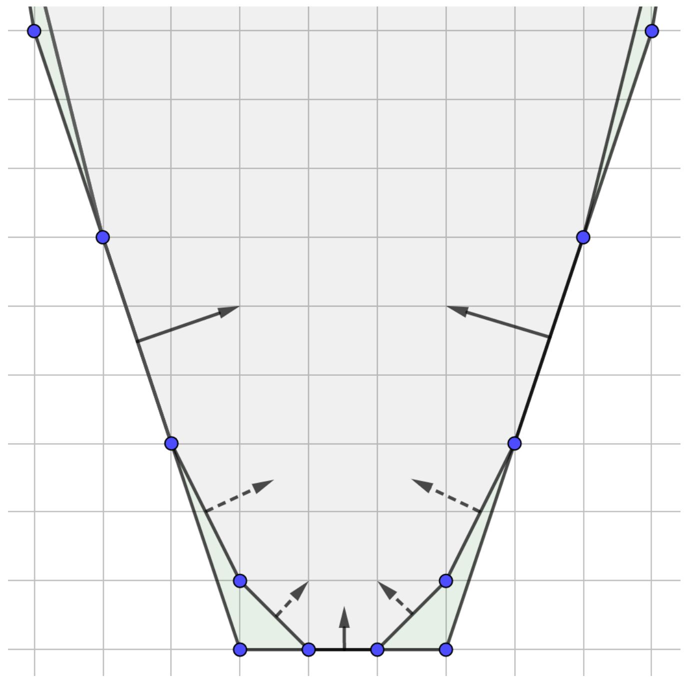
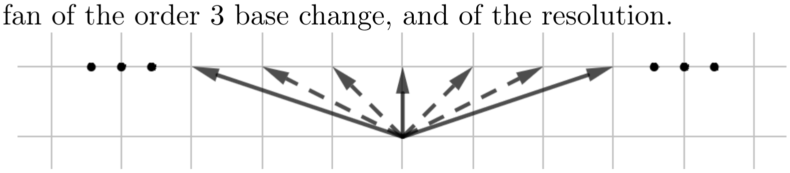


FIGURE 11. Above: order 3 base change of the Tate curve, with polarization $\mathcal{O}(3)$, and resolution. Following Remark 3.38, grid points are $(\frac{1}{3}\mathbb{Z})^2$. Below: normal fan of the order 3 base change, and of the resolution.



To resolve the total space $X(b^{(2)})$, we take the minimal resolution $\widetilde{X} \to X(b^{(2)})$, which resolves the A_2-singularity to a chain of two (-2)-curves $E_1 + E_2$. Then the central fiber $\widetilde{X} \to \Delta$ is an I_3-type Kodaira fiber, i.e. a wheel of $\mathbb{P}^1$s of length 3.

To realize this resolution as a *polarized* resolution, first, we pull back $3L$ to $\widetilde{X}$. It has multidegree $(3, 0, 0)$ on the three components of the wheel. Now, we twist, defining

$$\widetilde{L} \coloneqq 3L - E_1 - E_2.$$

The resulting line bundle has multidegree $(1, 1, 1)$ on the wheel, and the overgraph of the Mumford degeneration defining $(\widetilde{X}, \widetilde{L}) \to \Delta$ is shown in gray in Figure 11.

On the $3\mathbb{Z}$-prequotient, this is the usual blow-up operation on polytopes of polarized toric varieties, which cuts a corner off the polytope, whose size depends on the chosen polarization on the blow-up. The cut corners are depicted in green in Figure 11. The normal fans are depicted in the bottom of Figure 11. We see that, indeed the normal fan for $\widetilde{X} = X(3 \mid \frac{1}{3}b^{(1)}(3x))$ is a $3\mathbb{Z}$-invariant refinement of the normal fan to $X(b^{(2)})$. Thus, there is a $3\mathbb{Z}$-equivariant

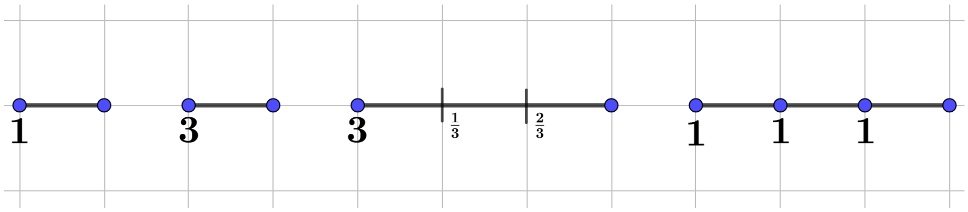


FIGURE 12. Left-to-right: the Tate curve, the order 3 base change of the Tate curve, the order 3 base change of the Tate curve plus 3rd Veronese embedding, and its polarized resolution. Integers at blue vertices are the bending parameters.

toric morphism between the corresponding toric varieties, descending on quotients to give the minimal resolution.

In terms of $\mathbb{T}^1$ and the bending loci, the various operations are depicted in Figure 12.

4. Regular matroids

We now discuss the construction of Mumford degenerations associated to regular matroids.

4.1. **Matroids, graphs, and quadratic forms.**

Definition 4.1. A *matroid* $\underline{R} = (\underline{R}, E)$ is a finite set E, together with a collection $\underline{R}$ of *independent subsets* of E, satisfying the following axioms:

(1) The empty set is independent.
(2) Any subset of an independent set is independent.
(3) If $I, J \subset E$ are independent sets with $|I| > |J|$, then there is an element $i \in I \setminus J$ for which $J \cup \{i\}$ is independent.

The set E is called the *ground set* of $\underline{R}$.

These axioms encapsulate the concept of linear independence of a collection of vectors in a vector space. A *basis* is a maximal independent subset $E' \subset E$, and a *circuit* is a minimal dependent set. Note that all bases of $\underline{R}$ have the same cardinality by Def. 4.1(3); this cardinality is called the *rank* of $\underline{R}$.

The *dual matroid* $\underline{R}^*$ is a matroid on the same ground set E, whose bases are the complements of bases of $\underline{R}$. A circuit of $\underline{R}^*$ is a *cocircuit* of $\underline{R}$.

Definition 4.2. A *realization* of $\underline{R}$ over the field $\mathbb{F}$ is a map $\phi\colon E \to \mathbb{F}^g$ to an $\mathbb{F}$-vector space for which the independent sets $E' \subset E$ are exactly those for which $\{\phi(i)\}_{i\in E'}$ are linearly independent. A matroid $\underline{R}$ is *regular* if it admits a realization over any field. An *integral realization* of a regular matroid is a map $\phi\colon E \to \mathbf{N}$ to a free $\mathbb{Z}$-module $\mathbf{N}$ which gives a realization of $\underline{R}$ upon base change $\mathbf{N} \otimes_{\mathbb{Z}} \mathbb{F}$ to any field $\mathbb{F}$.

We will always assume that $\phi(E)$ generates the lattice $\mathbf{N}$. In particular, the rank of $\underline{R}$ agrees with the rank of $\mathbf{N}$.

By a theorem of Tutte [63], every regular matroid can be defined by a *totally unimodular matrix*, that is, a matrix all of whose minors (in particular, all entries) have determinant in $\{\pm 1, 0\}$. Then, an integral realization of the matroid arises by considering the set of column vectors. More generally, any *unimodular matrix*—an integer entry matrix whose maximal minors have determinant in $\{\pm 1, 0\}$—defines an integral realization of a regular matroid [65, Ch. 3, Thm. 3.1.1]. Equivalently, the lattice spanned by any collection of columns is saturated.

Example 4.3. Let G be a graph and let $E = E(G)$ be its set of edges. Choose an orientation on the edges. We have an inclusion $H_1(G, \mathbb{Z}) \subset \mathbb{Z}^E$ as every homology class $\gamma \in H_1(G, \mathbb{Z})$ can be viewed as a $\mathbb{Z}$-linear combination of directed edges.

Let e_i denote the basis vector of $\mathbb{Z}^E$ corresponding to the i-th edge and let $\widetilde{\mathbf{x}}_i \coloneqq e_i^\vee \in (\mathbb{Z}^E)^\vee$ be the corresponding coordinate function. By restriction, we get a linear function $\mathbf{x}_i \in H_1(G, \mathbb{Z})^\vee \simeq H^1(G, \mathbb{Z})$. The *cographic matroid* $M^*(G)$ of G, on the ground set E, has realization

$$\begin{aligned} E &\to H^1(G, \mathbb{Z}), \\ i &\mapsto \mathbf{x}_i. \end{aligned}$$

The *graphic matroid* $M(G)$ is the dual matroid, and has realization

$$\begin{aligned} E &\to \mathbb{Z}^E / H_1(G, \mathbb{Z}), \\ i &\mapsto \overline{e}_i \end{aligned}$$

where $\overline{e}_i$ is the image of e_i under the natural quotient map. Its rank is $|E(G)| - \mathrm{rk}(H_1(G, \mathbb{Z}))$. The graphic and cographic matroids of G are sometimes called the *cycle* and *bond matroids* of G, respectively, in the matroid literature.

Remark 4.4. Let $T \subset E$ be a spanning forest of G. Associated to T is a basis of $H_1(G, \mathbb{Z})$ indexed by the edges in $E \setminus T$: each edge $i \in E \setminus T$ completes a unique closed circuit C_i of the graph G whose edges lie in $T \cup \{i\}$. These closed circuits determine a $\mathbb{Z}$-basis of $H_1(G, \mathbb{Z})$ and form the circuits of the graphic matroid $M(G)$.

Let $g = g(G)$ be the genus of the graph and $k = |E|$ be the number of edges. Then, in this basis, the integral realization of the cographic matroid $M^*(G)$ in Example 4.3 is a $g \times k$ matrix of the form $M^*_G = (\mathrm{Id}_g \,|\, P)$ where P is a matrix of 0's and ± 1's, and whose i-th row consists of the directed edges of T involved in the circuit C_i.

Example 4.5. If G is taken to be the theta graph (see Figure 13), with $(g, k) = (2, 3)$, and the spanning tree T is $\{e_3\} \subset E$, then

$$M^*_G = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

because the circuits completed by e_1 and e_2 are, respectively, $e_1 + e_3$ and $e_2 + e_3$.

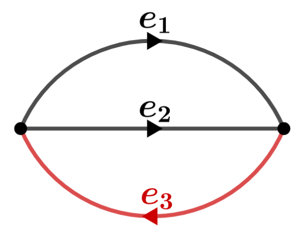


FIGURE 13. The theta graph, with spanning tree in red.

Definition 4.6. Let $i \mapsto \mathbf{x}_i \in \mathbf{N} \simeq \mathbf{M}^\vee$ be an integral realization of a regular matroid $\underline{R}$. Then the associated *matroidal cone* $\mathbb{B}_{\underline{R}}$ is the $\mathbb{R}_{\geq 0}$-span of $\mathbf{x}_i^2 \in \mathrm{Sym}^2\mathbf{M}^\vee$.

For example, let G be a graph. Its *cographic cone* $\mathbb{B}_{M^*(G)}$ is the cone of symmetric, positive semi-definite bilinear forms on $H_1(G, \mathbb{Z})$ given by

$$\begin{aligned}\mathbb{B}_{M^*(G)} &:= \mathbb{R}_{\geq 0}\{\mathbf{x}_i^2 : \ i \in E(G)\} \\ &= \{M_G^* D (M_G^*)^T \ : \ D \text{ diagonal with } D_i \geq 0\} \subset \mathcal{P}_g^+.\end{aligned}$$

See Alexeev-Brunyate and Melo–Viviani for analyses of which matroidal cones appear in various toroidal compactifications of $\mathcal{A}_g$ [5, 41].

Example 4.7 (Seymour-Bixby [58, 13])**.** Consider the totally unimodular matrix

$$R_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then, the columns of R_{10} define a regular matroid on 10 elements in $\mathbb{Z}^5$. We have already seen this matroid, which can be identified with the 10 vanishing cycles $\gamma_i \in \mathrm{gr}^W_{-2} V_{\mathbb{Z}} \simeq \mathbb{Z}^5$ of the nodes of the Segre cubic threefold, $V_{\mathbb{Z}} = H_3(Y_*, \mathbb{Z})(-1)$, see Example 2.16.

The associated matroidal cone is

$$\begin{aligned}\mathbb{B}_{\underline{R}_{10}} = \mathbb{R}_{\geq 0}\{&\mathbf{x}_1^2,\, \mathbf{x}_2^2, \mathbf{x}_3^2,\, \mathbf{x}_4^2,\, \mathbf{x}_5^2,\, (\mathbf{x}_5 - \mathbf{x}_1 + \mathbf{x}_2)^2,\, (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3)^2, \\ &(\mathbf{x}_2 - \mathbf{x}_3 + \mathbf{x}_4)^2,\, (\mathbf{x}_3 - \mathbf{x}_4 + \mathbf{x}_5)^2,\, (\mathbf{x}_4 - \mathbf{x}_5 + \mathbf{x}_1)^2\} \subset \mathrm{Sym}^2\, \mathbf{M}^\vee \otimes \mathbb{R}\end{aligned}$$

where $\mathbf{x}_i$ for $i = 1, \ldots, 5$ are the coordinates on $\mathbf{M} \simeq \mathbb{Z}^5$ given by the first five columns of R_{10}. A particularly nice realization of $\underline{R}_{10}$ over $\mathbb{F}_2^5$ is as the 10 vectors in $(\mathbb{F}_2)^{\oplus 6}/\mathbb{F}_2\langle 1, 1, 1, 1, 1, 1\rangle$ which have exactly three nonzero entries, cf. Example 2.16. In this realization, the full automorphism group $S_6 \simeq \mathrm{Aut}(\underline{R}_{10})$ is readily visible.

Remark 4.8. Examples 4.3 and 4.7 are essentially different: the matroid $\underline{R}_{10}$ is not isomorphic to $M(G)$ or $M^*(G)$, for any graph G, see e.g. [58].

Definition 4.9. Let $f^*\colon X^* \to Y^*$ be a smooth, projective family of PPAVs over a smooth quasiprojective base Y^*, and let $Y^* \hookrightarrow Y$ be an snc extension. We say that the morphism f^* is *matroidal* with respect to the extension Y if:

(1) the monodromy at the boundary $Y \setminus Y^*$ is unipotent, and
(2) the monodromy cones at all snc strata of $Y \setminus Y^*$ are matroidal cones.

Matroidal morphisms exist, in view of the following result:

Proposition 4.10. *Let $(\underline{R}, E)$ be a regular matroid of rank g on a k element set $E = \{1, \dots, k\}$, with integral realization $E \to \mathbf{N} = \mathbf{M}^\vee$, $i \mapsto \mathbf{x}_i$, and let $(r_1, \dots, r_k) \in \mathbb{N}^k$ be a vector of positive integers. Then there is a smooth projective family $f^*\colon X^* \to Y^*$ of g-dimensional PPAVs over a smooth quasiprojective base of dimension k, such that the following hold:*

(1) There is a smooth extension $Y^* \subset Y$ with snc boundary divisor $D = Y \setminus Y^*$ and an embedded polydisc $\Delta^k \subset Y$ such that the restriction of D to Δ^k agrees with the union $\{u_1 \cdots u_k = 0\}$ of the coordinate hyperplanes.
(2) Consider the base change $X^*_{(\Delta^*)^k} \to (\Delta^*)^k$ and let $t \in (\Delta^*)^k$. Then there is an isomorphism $\mathrm{gr}_0^W H_1(X_t, \mathbb{Z}) \simeq \mathbf{M}$ under which the monodromy bilinear form B_i around the i-th coordinate hyperplane (Def. 2.6) is given by $r_i \mathbf{x}_i^2$.

Proof. Apply Corollary 2.30 to the symmetric bilinear forms $B_i = r_i \mathbf{x}_i^2 \in \mathcal{P}_g^+ \cap \mathrm{Sym}_{g\times g}(\mathbb{Z})$. □

In the following sections, we will study regular extensions $f\colon X \to Y$ of matroidal morphisms.

4.2. Mumford degenerations associated to regular matroids.

Definition 4.11. A *hyperplane arrangement* is a finite collection $\{\overline{H}_i\}_{i\in I}$ of torsion translates $\overline{H}_i \subset \mathbb{T}^g = \mathbf{M}_\mathbb{R}/\mathbf{M}$ of codimension 1 subtori. Equivalently, it is a finite collection $\{H_i\}_{i\in I}$ where H_i is the union of all $\mathbf{M}$-translates of an affine linear hyperplane in $\mathbf{M}_\mathbb{R}$ defined over $\mathbb{Q}$.

Let $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbf{N}$ be a finite collection of vectors, $\operatorname{rank} \mathbf{N} = g$, giving an integral realization of a regular matroid $\underline{R}$. Each $\mathbf{x}_i$ defines a family of parallel hyperplanes

$$H_i \coloneqq \{\mathbf{m} \in \mathbf{M}_\mathbb{R} \,:\, \mathbf{x}_i(\mathbf{m}) \in \mathbb{Z}\} \subset \mathbf{M}_\mathbb{R}. \tag{27}$$

Then $\{H_1, \dots, H_k\}$ defines a hyperplane arrangement, see Definition 4.11. Moreover, H_i is the bending locus of the convex $\mathbb{Z}$PL function b_i on $\mathbf{M}_\mathbb{R}$ that satisfies

$$b_i(\mathbf{m}) = \frac{\mathbf{x}_i(\mathbf{m})^2 - \mathbf{x}_i(\mathbf{m})}{2} \text{ for } \mathbf{m} \in \mathbf{M}.$$

Note that b_i descends, as a convex section $\bar{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$. The regularity of the matroid implies that the b_i are dicing, i.e. the polytopes cut by the union of hyperplanes $H_{\underline{R}} \coloneqq \bigcup_{i=1}^k H_i$ have integral vertices. See Erdahl–Ryshkov [28].

Definition 4.12. We define the *matroidal Mumford construction* $X(\underline{R}) \to \Delta^k$ to be the Mumford construction $X(b_1, \dots, b_k) \to \Delta^k$ associated to the collection $(\mathbb{T}^g, \bar{b}_1, \dots, \bar{b}_k)$ of sections $\bar{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ above, defined by the regular matroid $\underline{R}$.

Then k is the size of the ground set of $\underline{R}$ while the dimension g of the fibers is the rank of $\underline{R}$. By construction, the monodromy cone of the matroidal Mumford construction on $\underline{R}$ is given by the matroidal cone $\mathbb{B}_{\underline{R}} \subset \mathcal{P}_g^+$ because the bilinear form B_i associated to b_i is $B_i = \mathbf{x}_i^2$.

Example 4.13 (Cographic matroids)**.** We have implicitly seen an important matroid, realized by the vanishing cycles associated to a nodal projective curve C_0 with k nodes, as in Example 2.15. We aim to

(1) identify the matroid realized by the vanishing cycles as the cographic matroid $M^*(G)$ where $G := H_1(\Gamma(C_0), \mathbb{Z})$ is the dual graph of the nodal curve, and
(2) under the simplifying assumption that the normalized components of C_0 have genus zero, use this identification and Construction 4.12 to compactify the relative Jacobian of the universal deformation $\pi\colon \mathcal{C} \to \mathrm{Def}_{C_0} \simeq \Delta^{3g-3} \simeq \Delta^k \times \Delta^{(3g-3)-k}$.

We begin with (1). Let C_t be a smooth fiber nearby C_0. Let $E = \{1, \dots, k\}$ be the set of nodes of C_0 and let $\gamma_1, \dots, \gamma_k \in H_1(C_t, \mathbb{Z})$ be the corresponding vanishing cycles, unique up to sign. This realizes a matroid on the ground set E. A choice of sign for each γ_i is equivalent to a choice of orientation of the edges of $G = \Gamma(C_0)$. Using the intersection pairing and Poincaré duality on C_t, we may view each γ_i as a linear form on $H_1(C_t, \mathbb{Z}) \simeq H_1(JC_t, \mathbb{Z})$. This linear form vanishes on W_{-1} and hence descends to a linear form on $\mathrm{gr}_0^W H_1(C_t, \mathbb{Z}) \simeq \mathrm{gr}_0^W H_1(JC_t, \mathbb{Z})$.

We also have an identification $\mathrm{gr}_0^W H_1(C_t, \mathbb{Z}) \simeq H_1(\Gamma(C_0), \mathbb{Z})$; thus γ_i is identified with the linear form on $H_1(G, \mathbb{Z})$ giving the coordinate $i \mapsto \mathbf{x}_i = e_i^\vee \in H^1(G, \mathbb{Z})$ of the oriented edge $e_i \in E(G)$. So the matroid realized by the k vanishing cycles in C_t is isomorphic to the cographic matroid $M^*(G)$. Conversely, the cographic matroid $M^*(G)$ associated to any graph G arises this way, because we can construct a nodal projective curve C_0 whose dual complex is G.

We now compactify the relative Jacobian fibration $J\pi^\circ\colon J\mathcal{C}^\circ \to (\Delta^*)^k \times \Delta^{(3g-3)-k}$ of the punctured family $\pi^\circ\colon \mathcal{C}^\circ \to (\Delta^*)^k$ of smooth curves, as in (2). So take $\mathbf{M} = H_1(G, \mathbb{Z})$ and $\mathbf{N} = H^1(G, \mathbb{Z})$ and apply the universal form of the matroidal Mumford construction of Definition 4.12. If no $\mathbf{x}_i = 0$ nor $\mathbf{x}_i = \pm\mathbf{x}_j$ for $i \neq j$ (in matroidal language: $M^*(G)$ contains no circuits of length ≤ 2), we get an extension

$$X^{\mathrm{univ}}(M^*(G)) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{M^*(G)}}$$

of the universal family. Otherwise, we only get a degeneration

$$X_\circ^{\mathrm{univ}}(M^*(G)) \to (\Delta^k)^{\mathrm{univ}}$$

where $(\Delta^k)^{\mathrm{univ}} \to (\mathbb{C}^*)^{\binom{g+1}{2}-\ell}$ is a k-dimensional polydisk bundle for some $\ell < k$, for which the classifying map $(\Delta^k)^{\mathrm{univ}} \to \mathcal{A}_g$ loses dimension. So assume the former. It follows then from

Proposition 2.27 that the Torelli map extends to a morphism

$$\mathrm{Def}_{C_0} \simeq \Delta^k \times \Delta^{(3g-3)-k} \hookrightarrow \widetilde{\mathcal{A}}_g^{\mathbb{B}_{M^*(G)}}.$$

Then the pullback of $X^{\mathrm{univ}}(M^*(G))$ defines an extension of the relative Jacobian fibration

$$J\pi\colon JC \to \Delta^k \times \Delta^{(3g-3)-k}$$

where $\{0\} \times \Delta^{(3g-3)-k} \to \{0\} \times (\mathbb{C}^*)^{\binom{g+1}{2}-k}$ maps the locally trivial deformations of C_0 into the deepest toroidal stratum $\widetilde{\mathcal{A}}_g^{\mathbb{B}_{M^*(G)}}$.

Example 3.35 is an example of a matroidal Mumford construction on the cographic matroid of the theta graph (Example 4.5), where $C_0 = \mathbb{P}^1 \cup_{\{0,\,1,\,\infty\}} \mathbb{P}^1$ is the union of two smooth rational curves along three points. The dual graph $\Gamma(C_0)$ is the theta graph, as depicted in Figure 13.

Example 4.14 (Matroidal Mumford degeneration on $\underline{R}_{10}$)**.** Another example comes from the Seymour–Bixby matroid $\underline{R}_{10}$ (Example 4.7) which gives a degeneration $X(\underline{R}_{10}) \to \Delta^{10}$ of PPAVs of dimension 5 over a 10-dimensional polydisk. To produce a universal degeneration whose monodromy cone is $\mathbb{B}_{\underline{R}_{10}}$ we must twist by $a \in (\mathbb{C}^*)^5$ (here $5 = 15 - 10$ and $15 = \dim \mathcal{A}_5$). The resulting universal Mumford degeneration is an extension of the universal family

$$X^{\mathrm{univ}}(\underline{R}_{10}) \to \widetilde{\mathcal{A}}_5^{\mathbb{B}_{\underline{R}_{10}}}.$$

If $\pi\colon Y \to \mathrm{Def}_{Y_0} \simeq \Delta^{10}$ is the universal deformation of the Segre cubic threefold (Example 2.16), there is a morphism $\mathrm{Def}_{Y_0} \to \widetilde{\mathcal{A}}_5^{\mathbb{B}_{\underline{R}_{10}}}$ transversely slicing the deepest toroidal boundary stratum $(\mathbb{C}^*)^5$. The intersection is transverse because the monodromy about each coordinate hyperplane is $B_i = \mathbf{x}_i^2$—thus, in toroidal charts of $\widetilde{\mathcal{A}}_5^{\mathbb{B}_{\underline{R}_{10}}}$, the period map is approximated by a translate of a subtorus which transversely slices the deepest boundary stratum. The pullback of $X^{\mathrm{univ}}(\underline{R}_{10})$ defines an extension $IJ\pi\colon IJY \to \Delta^{10}$ of the relative intermediate Jacobian fibration $IJ\pi^\circ\colon IJY^\circ \to (\Delta^*)^{10}$ over the smooth locus.

Remark 4.15. By [41, Lem. 4.0.5, Cor. 4.0.6], we do not need to pass to some étale cover $\widetilde{\mathcal{A}}_g \to \mathcal{A}_g$ to produce the toroidal extension associated to a matroidal cone—up to quotienting by the group of symmetries of $\underline{R}$ and identifying some faces, every matroidal cone $\mathbb{B}_{\underline{R}}$ on a regular matroid $\underline{R}$ with no loops or parallel edges (*simple* regular matroids in *loc. cit.*) embeds into $(\mathcal{A}_g)_{\mathrm{trop}}$ as in Example 2.25. Indeed, there is a universal matroidal extension $\mathcal{A}_g \hookrightarrow \mathcal{A}_g^{\mathrm{mat}}$ whose fan is the union of all matroidal cones, on regular matroids of rank $\leq g$, with no loops or parallel edges. By [41, Thm. A], $\mathcal{A}_g^{\mathrm{mat}}$ is the toroidal extension of the maximal common subfan of the first and second Voronoi fans.

4.3. **Shifted and transversely shifted matroidal degenerations.** We describe here a modification of the matroidal Mumford construction of Definition 4.12 which produces a regular total space; this property is quite special, and under some additional hypotheses, characterizes Mumford degenerations with regular total space.

Construction 4.16 (Shifted matroidal degenerations)**.** Suppose that $E \to \mathbf{N}$, $i \mapsto \mathbf{x}_i$ gives an integral realization of a regular matroid. Consider the family of parallel hyperplanes $H_i^o = \{\mathbf{m} \in \mathbf{M}_\mathbb{R} \,:\, \mathbf{x}_i(\mathbf{m}) \in \mathbb{Z}\} \subset \mathbf{M}_\mathbb{R}$ for $i \in E$. Then, all H_i^o intersect at all lattices points $\mathbf{M}$, or in the quotient $\mathbb{T}^g = \mathbf{M}_\mathbb{R}/\mathbf{M}$, at the origin. Thus, we consider the shifted hyperplanes

$$H_i := \{\mathbf{m} \in \mathbf{M}_\mathbb{R} \,:\, \mathbf{x}_i(\mathbf{m}) \in \epsilon_i + \mathbb{Z}\}$$

for $\epsilon_i \in \frac{1}{d}\mathbb{Z}$. For sufficiently large d, it is possible to choose values of ϵ_i for which this shifted hyperplane arrangement $\{H_1, \dots, H_k\}$ satisfies the following additional property:

Definition 4.17. A hyperplane arrangement $\{H_i\}_{i\in I}$ is *transversal* if at any intersection point $p \in \bigcap_{i\in I} H_i$ the normal vectors of H_i for $i \in I$ are linearly independent.

Associated to H_i (transversal or not) we define a piecewise linear function $b_i \colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$ by the properties that:

(1) $b_i(m) := \frac{1}{2}(\mathbf{x}_i(\mathbf{m} - \mathbf{m}_0)^2 - \mathbf{x}_i(\mathbf{m} - \mathbf{m}_0))$ for any $\mathbf{m} \in \mathbf{m}_0 + \mathbf{M}$ where $\mathbf{m}_0 \in \mathbf{M}_\mathbb{R}$ is any point for which $\mathbf{x}_i(\mathbf{m}_0) = \epsilon_i$ and
(2) $\mathrm{Bend}(b_i) = H_i$ is the shifted family of hyperplanes.

The associated bilinear forms $B_i = \mathbf{x}_i^2$ are the same as for the unshifted case.

While the function b_i is not integer valued on integer points, it is $\frac{1}{d}$-integer valued on $\frac{1}{d}$-integral points. Thus, by Construction 3.38, we may take a Mumford degeneration

$$f\colon X(d \mid b_1, \dots, b_k) \to \Delta^k$$

whose monodromies B_i are the same as those of the matroidal Mumford degeneration. More generally, we have, by Construction 3.26, a universal form $X^{\mathrm{univ}}(d \mid b_1, \dots, b_k) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$. We may further generalize this set-up:

Definition 4.18. Let $\underline{R}$ be a regular matroid. A *shifted matroidal degeneration* on $\underline{R}$ is a Mumford construction $X(d \mid b_1, \dots, b_k) \to \Delta^k$ for which the bending locus of b_i is a union of parallel hyperplanes in $\mathbf{M}_\mathbb{R}$ whose primitive normal vectors $\mathbf{x}_i \in \mathbf{N}$ give an integral realization of $\underline{R}$. We furthermore call a shifted matroidal degeneration *transversely shifted* if

(1) the hyperplane arrangement $\{\mathrm{Bend}(b_i) \,:\, i = 1, \dots, k\}$ is transversal, and
(2) the bending parameter of b_i along each hyperplane is 1.

We remark that Definition 4.18 is well-defined, independent of the choice of primitive normal vector $\mathbf{x}_i \in \mathbf{N}$, which is only unique up to sign. Closely related constructions go by the name "multiplicative hypertoric variety" in more representation-theoretic literature, see especially [22, Sec. 8.3] (for the cographic case), generalizing the "additive" case [8, 12, 34]. ♣

See the right hand sides of Figures 14, 16 for examples of transversely shifted arrangements.

Notation 4.19. The bending locus $\mathrm{Bend}(b_i)$ can be viewed as a multiset $\{H_i^{(1)}, \dots, H_i^{(r_i)}\}$ of rational translates of the hyperplane normal to $\mathbf{x}_i$—a hyperplane H appears m times in the

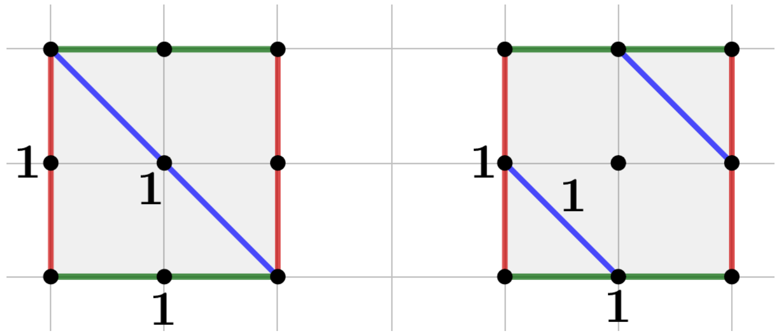


FIGURE 14. Left: 2nd Veronese embedding of the matroidal degeneration of the theta graph. Right: a shifted version. Grid points are $(\frac{1}{2}\mathbb{Z})^2$. The integers are the multiplicities of given hyperplane in the arrangement $\mathscr{H}$.

multiset if the bending parameter of b_i along H is $m \in \mathbb{N}$. Then, we assemble the hyperplane arrangement into a single symbol

$$\mathscr{H} := \{H_1^{(1)}, \dots, H_1^{(r_1)}, \dots, H_k^{(1)}, \dots, H_k^{(r_k)}\}$$

where $H_i^{(j)}$ is the set of $\mathbf{M}$-translates of a single hyperplane normal to $\mathbf{x}_i$. To indicate the relation to the regular matroid $\underline{R}$, we re-notate the Mumford construction of Definition 4.18, or the universal version, as

$$X(\underline{R}, \mathscr{H}) \to \Delta^k \quad \text{or} \quad X^{\text{univ}}(\underline{R}, \mathscr{H}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}.$$

Example 4.20 (Transversely shifted matroidal degeneration for the theta graph)**.** Beginning with the standard matroidal degeneration $X(M^*(G)) \to \Delta^3$ associated to the theta graph G (Example 3.35), take the degree 2 Veronese embedding. The resulting polyhedral decomposition of $\mathbb{T}^2$ and bending loci are depicted in the top left of Figure 14. Now consider the shifts of the families of hyperplanes H_1^o, H_2^o, H_3^o (of colors red, green, blue, respectively) of the standard arrangement for $M^*(G)$, by $(\epsilon_1, \epsilon_2, \epsilon_3) = (0, 0, \frac{1}{2})$. The result is a transversal arrangement $\mathscr{H} = \{H_1, H_2, H_3\}$ with $H_1 = H_1^o$, $H_2 = H_2^o$, $H_3 = H_3^o + (\frac{1}{2}, 0)$.

Applying Proposition 3.28, it can be seen that the central fiber of the left-hand Mumford degeneration in Figure 14 is the union of two copies of $\mathbb{P}^2$, both polarized by $\mathcal{O}_{\mathbb{P}^2}(2)$. Similarly, the central fiber of the righthand Mumford degeneration $X(M^*(G), \mathscr{H}) \to \Delta^3$ is the union of three surfaces: Two copies of $\mathbb{P}^2$, both polarized by $\mathcal{O}_{\mathbb{P}^2}(1)$, and a Cremona surface $V := Bl_{p_1,p_2,p_3}(\mathbb{P}^2)$, polarized by the anticanonical divisor $-K_V$.

Example 4.21. Consider the regular matroid defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

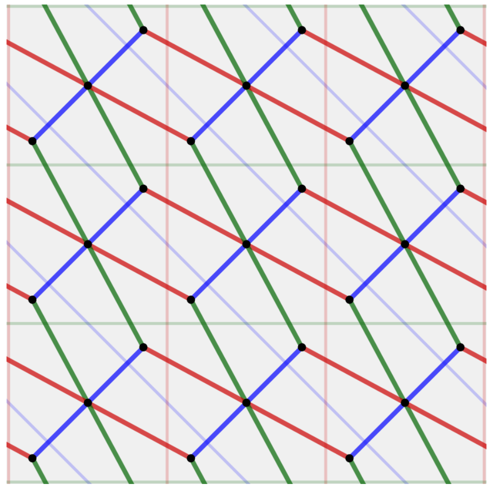


FIGURE 15. Dual complex (edges colored) of the righthand shifted arrangement in Figure 14. It is a tiling of a 2-torus by cubes.

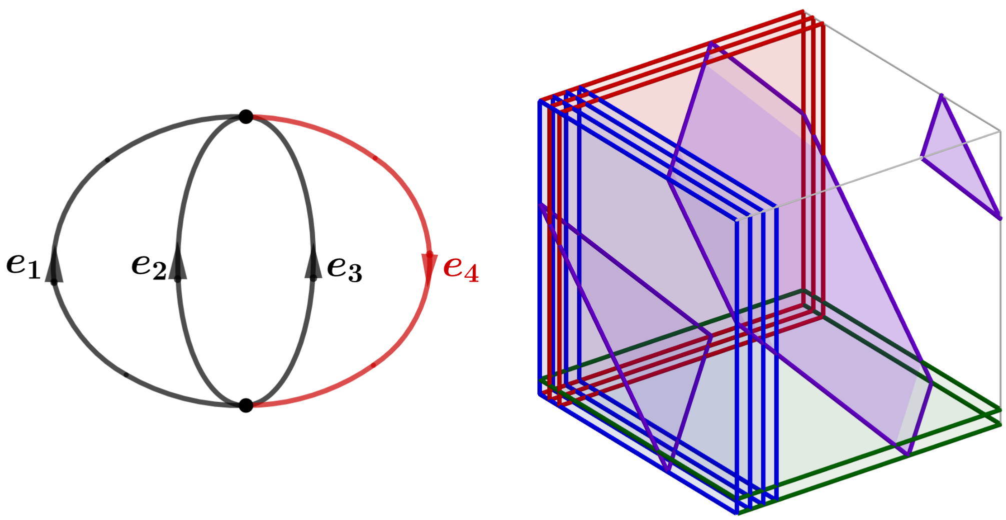


FIGURE 16. Left: Oriented genus 3 graph with spanning tree in red. Right: Transversely shifted hyperplane arrangement in $\mathbb{T}^3$.

It is the cographic matroid of the oriented graph G depicted in the left of Figure 16. Letting $\mathbf{x}_i \in H^1(G, \mathbb{Z})$ for $i = 1, 2, 3, 4$ be the linear forms corresponding to the four oriented edges of G, we form a transversely shifted matroidal degeneration $X(M^*(G), \mathscr{H})$ where

$$\mathscr{H} = \{H_1^{(1)}, H_1^{(2)}, H_1^{(3)}, H_2^{(1)}, H_2^{(2)}, H_2^{(3)}, H_2^{(4)}, H_3^{(1)}, H_3^{(2)}, H_4^{(1)}\}.$$

There are, respectively, 3, 4, 2, 1 hyperplanes perpendicular to $\mathbf{x}_1 = (1,0,0)$, $\mathbf{x}_2 = (0,1,0)$, $\mathbf{x}_3 = (0,0,1)$, $\mathbf{x}_4 = (1,1,1)$. These hyperplanes are, respectively, depicted in red, blue, green, and purple in Figure 16.

5. Nodal and semistable morphisms over higher-dimensional bases

Definition 5.1. Let Y be a smooth analytic space, and let $D \subset Y$ be an snc divisor, $D = \bigcup_i D_i$. Let $f\colon X \to Y$ be a morphism of analytic spaces. We say that f is

(1) *D-nodal* if for every point $p \in X$, there are analytic coordinates in which the morphism f is of the form
$$\prod_{i\in I}\{x_i y_i = u_i\} \times \Delta^{j+k} \to \prod_{i \in I} \Delta_{u_i} \times \Delta^j$$
where u_i are local equations for some components $D_i \subset D$, $i \in I$ and $\Delta^{j+k} \to \Delta^j$ is the projection to the first j coordinates,

(2) *nearly D-nodal* if we rather have a normal form of shape
$$\prod_{i\in I}\{x_i^{(1)} y_i^{(1)} = \cdots = x_i^{(n_i)} y_i^{(n_i)} = u_i\} \times \Delta^{j+k} \to \prod_{i \in I} \Delta_{u_i} \times \Delta^j,$$

(3) *D-semistable* if we have
$$\prod_{i\in I}\{x_i^{(1)} \cdots x_i^{(n_i)} = u_i\} \times \Delta^{j+k} \to \prod_{i \in I} \Delta_{u_i} \times \Delta^j.$$

In all three cases, if we furthermore have that the irreducible components $V_i \subset X_i$ of the generic fiber of f over each component of D_i are smooth, we use the term *strict*.

This definition works equally well in the algebraic category, replacing Δ with $\mathbb{A}^1$ and analytic-local charts with étale-local charts. In the cases where f is D-nodal or D-semistable, the total space X is smooth, but if some $n_i \geq 2$ for a nearly D-nodal morphism, then X is singular.

Remark 5.2. The notion of a D-semistable morphism is already known in the literature by the term *semistable morphism*, and when context is clear, we also drop the D. By Adiprasito–Liu–Temkin's resolution [2] of the conjecture of Abramovich–Karu [1], for every dominant morphism $f\colon X \to Y$, there is an alteration $Y' \to Y$ and a modification $X' \to X \times_Y Y'$ of the base change which is D-semistable, for the discriminant divisor D.

5.1. Mumford degenerations with nodal singularities.

Proposition 5.3. *Let $f\colon X(\underline{R}, \mathscr{H}) \to \Delta^k$ be a transversely shifted matroidal degeneration. Then, the morphism f has D-nodal singularities, where $D \coloneqq V(u_1 \cdots u_k) \subset \Delta^k$ is the union of the coordinate hyperplanes. In particular, $X(\underline{R}, \mathscr{H})$ is smooth. The same results hold for the morphism $f^{\mathrm{univ}}\colon X^{\mathrm{univ}}(\underline{R}, \mathscr{H}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ with D the toroidal boundary. Conversely, any Mumford degeneration with D-nodal singularities is a transversely shifted matroidal degeneration on some regular matroid $\underline{R}$.*

Proof. First we prove the forward direction.

Since the action of $d\mathbf{M}$ on the universal cover of the Mumford degeneration is free, it suffices to check the statement on this universal cover. We first show that every cone of the normal fan to Γ is a standard affine cone (i.e. integral-affine equivalent to $\mathbb{N}^i$ for some i).

The cones of the normal fan are in bijection with the faces of Γ. Let F be a polyhedral face in the decomposition $\bigcup_{i=1}^k H_i$ of $\mathbb{T}^g$. Then F also defines a face of Γ by evaluating $(b_1, \dots, b_k)$ on F. Conversely, all faces of Γ contain such a face in their closure. So it suffices to examine the normal fan of the faces adjacent to F.

By the transversality hypothesis, F is locally described as an intersection $\bigcap_{i \in I} H_i$ for which $\mathbf{x}_i \in \mathbf{N}$ for $i \in I$ are linearly independent. Furthermore, the normal vectors $\mathbf{x}_i$ for $i \in I$ generate a standard affine cone in $\mathbf{N}$, because $i \mapsto \mathbf{x}_i$ is an integral realization of a regular matroid, and so the sublattice generated by them is saturated.

By hypothesis, the bending parameter of b_i is 1 across any hyperplane H_i with normal vector $\mathbf{x}_i$. Thus, after an integral change of basis and translation of F to the origin, b_i is locally expressible as

$$b_i(\mathbf{m}) = \begin{cases} 0 & \text{if } \mathbf{x}_i(\mathbf{m}) \leq 0 \\ \mathbf{x}_i(\mathbf{m}) & \text{if } \mathbf{x}_i(\mathbf{m}) \geq 0 \end{cases} \tag{28}$$

on $\mathbf{M}_\mathbb{R}$ for $i \in I = \{1, \dots, k\}$, $k \leq g$. Combining all the above considerations, we deduce that the local monoid $\mathbf{M}_F$ of the face F is a product

$$\mathbf{M}_F = \prod_{i \in I} (\mathbb{Z}_{\geq 0})^2 \times \prod_{i \,\in \, \mathrm{Basis} \setminus I} \mathbb{Z} \times \prod_{E \setminus I} \mathbb{Z}_{\geq 0} \subset \mathbf{M} \times \mathbb{Z}^k \simeq \mathbb{Z}^{g+k}$$

where the first factors, indexed by $i \in I$, correspond to two vectors along the graph of b_i in the two (local) domains of linearity of (28), the second factors go along the face F, and the third factors, indexed by $E \setminus I$, are "vertical" faces, arising from the fact that we took the overgraph $\Gamma(b_1, \dots, b_k)|_F + (\mathbb{Z}_{\geq 0})^k$. Note that the dimensions of the factors add up to the correct value

$$2|I| + (g - |I|) + (|S| - |I|) = g + k.$$

The dual cone to $\mathbf{M}_F$ is isomorphic to $(\mathbb{Z}_{\geq 0})^{|I|+k}$.

We deduce that the cones of the normal fan are standard affine, and so X is smooth. Furthermore, the morphism to the fan $(\mathbb{R}_{\geq 0})^k$ is, on the first factors $(x, y) \mapsto x + y$, on the second factor is zero, and on the third factor, is an isomorphism to the coordinate axis indexed by the corresponding element of $E \setminus I$. We deduce that the morphism $\mathfrak{F}_F \to (\mathbb{R}_{\geq 0})^k$ is a product of node smoothings, with a smooth morphism, as in the definition of a D-nodal morphism. The second statement follows.

We now prove the reverse direction. The condition that $X(d \mid b_1, \dots, b_k) \to \Delta^k$ have D-nodal singularities implies that the polyhedral complex $\mathrm{Bend}(b_i)$ can only have codimension 1 faces. By convexity, we deduce that $\mathrm{Bend}(b_i)$ is a disjoint union of parallel hyperplanes H_i in $\mathbb{T}^g$. At any face F of $\bigcup_{i=1}^k \mathrm{Bend}(b_i)$ where these hyperplanes intersect, the normal vectors $\mathbf{x}_i$ to the hyperplanes must be a subset of a $\mathbb{Z}$-basis of $\mathbf{N}$ for the normal fan to be standard affine. It follows that any linearly independent collection of $\mathbf{x}_i$ generate a saturated sublattice of $\mathbf{N}$ and so $\mathbf{x}_i$ define a regular matroid $\underline{R}$. Furthermore, the fact that the normal vectors must be linearly independent proves that the H_i define a transversal arrangement.

The same results hold for $X^{\mathrm{univ}}(\underline{R}, \mathscr{H}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$, which over the neighborhood $(\Delta^k)^{\mathrm{univ}}$ of the boundary forms a locally trivial deformation of the degeneration $X(\underline{R}, \mathscr{H}) \to \Delta^k$. $\square$

Question 5.4. A 1-parameter semistable degeneration $f\colon X \to Y$ which is relatively K-trivial (i.e. $K_X \sim_f 0$) is called a Kulikov model, see Remark 3.5. Proposition 5.3 shows that it is natural to generalize this notion to a *multivariable Kulikov model*—a proper, semistable, relatively K-trivial morphism. It is unclear in what context they are guaranteed to exist. Proposition 5.3 gives nontrivial examples of such, for abelian varieties. Given a family of K-trivial varieties $f\colon X \to Y$, is there an alteration of the base $Y' \to Y$ and birational modification $f'\colon X' \to Y'$ of the base change, which is a multivariable Kulikov model? Do multivariable Kulikov models exist for families of K3 surfaces?

Proposition 5.5. *Suppose that every face F of a transversal arrangement $\mathscr{H}$ is embedded in $\mathbb{T}^g$ (as opposed to immersed) for all subsets $I \subset E$. Then, every stratum of $X = X(\underline{R}, \mathscr{H})$ is smooth. In particular, $f\colon X \to \Delta^k$ is strictly D-nodal. Thus, a transversely shifted matroidal degeneration is strictly D-nodal if and only if there are least two hyperplanes with normal vector $\mathbf{x}_i \in \mathbf{N}$ for all $i = 1, \dots, k$.*

Proof. It follows from the hypothesis and Proposition 3.28 that if the $d\mathbf{M}$-action on the universal cover $X(\mathcal{S})$ of the Mumford degeneration identifies two points $p, q \in X(\mathcal{S})$ lying on the same smooth toric stratum, then p, q must lie in a subtorus $(\mathbb{C}^*)^k$ being quotiented to an abelian k-fold. The first part of the proposition follows.

To show the second part: Suppose that there are $r_i \geq 2$ hyperplanes with normal vector $\mathbf{x}_i$. Let I_0 be a subset of a basis, say $I_0 = \{1, \dots, h\} \subset \{1, \dots, g\}$ for which the span of $\mathbf{x}_i$ for $i \in I$ is generated by the $\mathbf{x}_i$ for $i \in I_0$. Then $\bigcup_{H \in \mathscr{H}} H$ is, combinatorially, a tiling of $\mathbb{T}^g$ by the product of a subtorus $\mathbb{T}^{g-h} \subset \mathbb{T}^g$ with some polyhedral subdivision of the tiling of $\mathbb{T}^h$ by cubes of size $1/r_1 \times \cdots \times 1/r_h$. All such cubes are embedded in $\mathbb{T}^h$ once $r_i \geq 2$. $\square$

Remark 5.6. Let $\mathscr{H}$ define a transversal arrangement. Then, any small rational perturbation of the $H \in \mathscr{H}$ which keeps the combinatorics of the intersection complex $\bigcup_{H \in \mathscr{H}} H \subset \mathbb{T}^g$ constant produces an isomorphic degeneration $X(\underline{R}, \mathscr{H}) \to \Delta^k$, since the normal fan is unchanged by such a perturbation. Thus, the only difference between these two Mumford constructions is the choice of polarization on the total space. More generally:

Proposition 5.7. *For any arrangement $\mathscr{H}$, and any sufficiently small perturbation $\mathscr{H}'$ of $\mathscr{H}$, there is a birational morphism $X(\underline{R}, \mathscr{H}') \to X(\underline{R}, \mathscr{H})$ over Δ^k.*

Proof. The proposition follows from the fact that the normal fan associated to $\mathscr{H}'$ is a refinement of that for $\mathscr{H}$—all domains of linearity of the PL function $(b_1, \dots, b_k)\colon \mathbf{M}_{\mathbb{R}} \to \mathbb{R}^k$ bending along $\mathscr{H}$ "persist" (up to a small deformation) as domains of linearity for the PL function $(b'_1, \dots, b'_k)\colon \mathbf{M}_{\mathbb{R}} \to \mathbb{R}^k$ bending along $\mathscr{H}'$. Hence, any normal cone to a face of the polytope Γ is a union of normal cones of the corresponding faces of Γ'. □

Corollary 5.8. *A small, transversal perturbation $\mathscr{H}'$ of the hyperplane arrangement (27) defining the standard matroidal Mumford construction $X(\underline{R}) \to \Delta^k$ (Def. 4.12) defines a projective resolution of singularities $X(\underline{R}, \mathscr{H}') \to X(\underline{R})$, which is D-nodal over Δ^k.*

Proof. The corollary follows from Propositions 5.7 and 5.3. □

5.2. **Weight filtration of a semistable morphism.** Throughout this section, suppose that $f\colon X \to Y$ is a (strict) D-semistable, proper morphism, for an snc pair (Y, D), with X Kähler. We fix a point $0 \in D$ and denote by D_I^o the open snc stratum containing 0. Assume without loss of generality that $I = \{1, \dots, k\}$. We also fix a base point $t \in Y \setminus D$ near 0.

Proposition 5.9. *Let $f\colon X \to Y$ be a (strict) D-semistable morphism for an snc pair (Y, D) and let $C \to (Y, D)$ be a pointed curve which transversely intersects the open stratum of a component $D_i \subset D$. Then the base change $X \times_Y C \to C$ is a (strict) semistable degeneration.*

Proof. This follows immediately from the normal form (3). □

Consider the inclusion of the nearby fiber $X_t \hookrightarrow X_{\Delta^I}$ into the restriction of $X \to Y$ to a polydisk $\Delta^I \ni 0$ transversely slicing the snc stratum $D_I^o \ni 0$. We claim:

Proposition 5.10. *There is a deformation-retraction $c\colon X_{\Delta^I} \to X_0$. The fibers of $c_t := c|_{X_t}$ are real tori; more precisely, if $p \in X_0$ lies in a product of snc strata $p \in \prod_{i \in I} V(x_i^{(1)}, \cdots, x_i^{(n_i)})$, cf. (3), then the fiber of c_t is $c_t^{-1}(p) = \prod_{i \in I} (S^1)^{n_i - 1}$ where $(S^1)^{n_i-1} \subset X_t$ is the vanishing torus of the semistable degeneration $x_i^{(1)} \cdots x_i^{(n_i)} = u_i$.*

It would be natural to call the retraction c the "multivariable Clemens collapse", in analogy with the Clemens collapse of a 1-parameter semistable degeneration, as in [19, Thm. 5.7], [52, Sec. 2.3], [53, Prop. C.11].

Proof of Proposition 5.10. Consider first a 1-parameter semistable degeneration over Δ_u as for the usual Clemens collapse. Near an snc stratum of the fiber, one defines c as the deformation-retraction of $\{x^{(1)} \cdots x^{(n)} = u\} \subset \Delta^n \times \Delta_u$ to $\{x^{(1)} \cdots x^{(n)} = 0\} \subset \Delta^n \times \{0\}$ given by keeping the arguments of the complex numbers $x^{(j)}$ constant and linearly decreasing the absolute values $|x^{(j)}|$ until one of these absolute values equals zero. These local deformation-retractions may be patched via partitions of unity, cf. [19, p. 236], to give a piecewise smooth retraction of the total space of the semistable degeneration onto its central fiber.

The same procedure works in the D-semistable case: We define a deformation-retraction c of
$$\prod_i \{x_i^{(1)} \cdots x_i^{(n_i)} = u_i\} \subset \Delta^n \times \prod_i \Delta_{u_i}$$
onto $\prod_i \{x_i^{(1)} \cdots x_i^{(n_i)} = 0\} \subset \Delta^n \times \{0\}$ where $n = \sum n_i$. It is the product of the 1-variable retractions, linearly decreasing the absolute values of $x_i^{(j)}$ until one of them equals zero, for each i. Note that c fibers over the deformation-retraction of the base $\prod_i \Delta_{u_i}$ to the origin, which radially contracts each coordinate u_i until it equals zero.

The fiber $c_t^{-1}(p)$ over the origin p of the given chart is a product of tori—each torus factor $(S^1)^{n_i-1}$ is given by
$$|x_i^{(1)}| = \cdots = |x_i^{(n_i)}| = |u_i(t)|^{1/n_i}.$$
Since one linearly decreases the absolute values $|x_i^{(j)}|$ for $j = 1, \ldots, n_i$ simultaneously, they all hit the value zero at the same time. Finally, one patches the local retractions thus defined over products of snc strata, in a manner similar to [19]. □

Denote the local monodromy operators on (co)homology about the component $D_i \ni 0$ by
$$T_i \colon H^q(X_t, \mathbb{Z}) \to H^q(X_t, \mathbb{Z}) \text{ or } H_q(X_t, \mathbb{Z}) \to H_q(X_t, \mathbb{Z}).$$
A semistable degeneration over a curve has unipotent monodromy, so by Proposition 5.9, the T_i are commuting unipotent operators. Let
$$N_i := \log T_i := (T_i - \mathrm{Id}) - \tfrac{1}{2}(T_i - \mathrm{Id})^2 + \tfrac{1}{3}(T_i - \mathrm{Id})^3 - \cdots$$
be their nilpotent logarithms. Any linear combination $N := \sum a_i N_i$ for $a_i \in \mathbb{N}$ positive integers, defines the same weight filtration $W_\bullet$ on $H^q(X_t, \mathbb{Z})$ or $H_q(X_t, \mathbb{Z})$ with weights lying between 0 and $2q$, resp. $-2q$ and 0. Here we use that X is Kähler, so that that $W_\bullet$ is the weight filtration of the limit mixed Hodge structure.

Proposition 5.11. *Let $X \to Y$ be a strict D-semistable degeneration, X Kähler, $0 \in Y$ a point and $t \in Y$ a nearby point. There is a canonical specialization map* $\mathrm{sp} \colon H_q(X_t, \mathbb{Z}) \to H_q(\Gamma(X_0), \mathbb{Z})$ *for all q. Furthermore,* sp *is surjective when $q = 1$.*

Proof. Proposition 5.10 produces a map $c_t \colon X_t \to X_0$. Our goal is now to define a homotopy equivalence $\widetilde{X}_0 \to X_0$ from a new topological space $\widetilde{X}_0$, that admits a map $\widetilde{X}_0 \to \Gamma(X_0)$, so that we have the following diagram:
$$X_t \to X_0 \leftarrow \widetilde{X}_0 \to \Gamma(X_0);$$
our map sp will be the composition $H_1(X_t, \mathbb{Z}) \to H_1(X_0, \mathbb{Z}) \xleftarrow{\sim} H_1(\widetilde{X}_0, \mathbb{Z}) \to H_1(\Gamma(X_0), \mathbb{Z})$.

The topological space $\widetilde{X}_0$ is built as follows. Let $X_0 = \bigcup V_j$ be the decomposition of X_0 into its irreducible components and V_J denote (an irreducible component of) $\bigcap_{j \in J} V_j$. Each stratum $V_J \subset X_0$ is locally a product of snc strata, with local form
$$\textstyle\prod_{i \in I} \{x_i^{(1)} = \cdots = x_i^{(n_i)} = 0\} \subset \prod_{i \in I} \{x_i^{(1)} \cdots x_i^{(n_i)} = 0\},$$

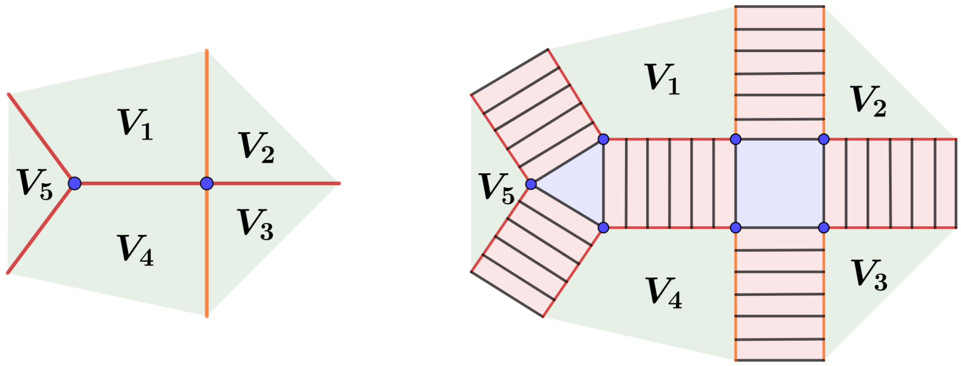


FIGURE 17. Left: The central fiber of a semistable degeneration $X \to \Delta_{u_1} \times \Delta_{u_2}$ with five components $V_i \subset X_0$, $i = 1,2,3,4,5$ of the central fiber, in green. Double loci extending over $V(u_1)$ in red and double loci over $V(u_2)$ in orange. Local equation of the smoothing of the lefthand triple locus V_{145} (in blue) is $x^{(1)}y^{(1)}z^{(1)} = u_1$ and local equation of the smoothing of the righthand codimension 2 stratum V_{1234} (in blue) is $\{x^{(1)}y^{(1)} = u_1\} \times \{x^{(2)}y^{(2)} = u_2\}$. Right: Topological space $\widetilde{X}_0$ with double loci V_{12}, V_{23}, V_{34}, V_{45}, V_{51} replaced with 1-simplex bundles Σ_{12}, Σ_{23}, Σ_{34}, Σ_{45}, Σ_{51} and with V_{145} and V_{1234} replaced, respectively, with 2-simplex and $(1,1)$-polysimplex (i.e. square) bundles Σ_{145} and Σ_{1234}. The dual complex $\Gamma(X_0)$ is the blue triangle glued to the blue square.

see Definition 5.1(3). Thus, the local dual complex of X_0 at any point $v \in V_J^o$ in the open snc stratum is a product of $(n_i - 1)$-simplices, and these local dual complexes form a product-of-simplices, i.e. polysimplex bundle $\Sigma_J \to V_J$. Since we assume that $f\colon X \to Y$ is strict D-semistable, the polysimplex bundle Σ_J is in fact trivial: $\Sigma_J \simeq_{\text{homeo}} V_J \times \prod_i \sigma_{n_i - 1}$. We define

$$\widetilde{X}_0 \coloneqq \bigsqcup\nolimits_J \Sigma_J / \sim$$

where $\sim$ is the equivalence relation given by the inclusion of $\Sigma_J|_{V_{J'}} \hookrightarrow \Sigma_{J'}$ corresponding to the face inclusion of the product of simplices $\prod \sigma_{n_i-1}$ corresponding to the inclusion of subsets $J \subset J'$. See Figure 17.

Then $\widetilde{X}_0$ has a homotopy equivalence to X_0 by decreasing the proportions of the polysimplices from side length 1 to 0. Furthermore, there is a natural contraction map

$$\mu\colon \widetilde{X}_0 \to \Gamma(X_0)$$

given by collapsing each open snc stratum to a point, which collapses the polysimplex bundle Σ_J to a polysimplex $\prod_i \sigma_{n_i-1}$. For instance, in righthand side of Figure 17, the components are contracted to points, and the interval bundles over double loci are contracted to intervals.

Noting that $H_q(X_0, \mathbb{Z}) \simeq H_q(\widetilde{X}_0, \mathbb{Z})$ by the homotopy equivalence $\widetilde{X}_0 \to X_0$ we may then define $\mathrm{sp} \coloneqq \mu_* \circ (c_t)_*\colon H_q(X_t, \mathbb{Z}) \to H_q(\Gamma(X_0), \mathbb{Z})$.

To prove surjectivity of sp when $q = 1$, let $\alpha_0 \in H_1(\Gamma(X_0), \mathbb{Z})$. Then, we may lift α_0 to an element $\alpha \in H_1(\Gamma^{[1]}(X_0), \mathbb{Z})$ as the map $H_1(\Gamma^{[1]}(X_0), \mathbb{Z}) \to H_1(\Gamma(X_0), \mathbb{Z})$ is surjective for any polyhedral complex.

Starting with the vertices $v_j \in \alpha^{[0]}$, we fix a lift $\widetilde{v}_j \in V_j^o$ in the open stratum of the corresponding component. For an edge $e_{jj'} \in \alpha^{[1]}$ connecting vertices v_j and $v_{j'}$, we lift to a path $\widetilde{e}_{jj'}$ in X_0 connecting $\widetilde{v}_j$ and $\widetilde{v}_{j'}$ and crossing the open stratum $V_{jj'}^o$. This produces a lift $\widetilde{\alpha}$ of α_0 to a closed singular 1-chain in X_0. To further lift to $H_1(X_t, \mathbb{Z})$, consider the inverse image

$$\alpha_t \coloneqq c_t^{-1}(\widetilde{\alpha} \cap X_0^{\mathrm{reg}})$$

of the intersection of $\widetilde{\alpha}$ with the regular locus.

Consider the point $p_{jj'} = \widetilde{e}_{jj'} \cap V_{jj'}^o$ where the edge $\widetilde{e}_{jj'}$ crosses a double locus. There are two limit points of α_t on the circle $c_t^{-1}(p_{jj'})$. We may connect these limiting points by an arc of the circle so as to lift the corresponding path $\widetilde{e}_{jj'}$ into X_t. The result is a closed 1-chain in X_t whose homology class maps to α_0 under sp. □

Proposition 5.12. *Let $f\colon X \to Y$ be a strict D-semistable morphism over an snc pair (Y, D), X Kähler, and let $0 \in Y$. We have a canonical isomorphism $\mathrm{gr}_0^W H_1(X_t, \mathbb{Z}) \simeq H_1(\Gamma(X_0), \mathbb{Z})$.*

Proof. By the definition of the integral weight filtration, we have

$$\mathrm{gr}_0^W H_1(X_t, \mathbb{Z}) = H_1(X_t, \mathbb{Z}) / \ker(N)$$

and $\ker(N) = \bigcap_{i \in I} \ker(N_i)$. By the second part of Proposition 5.11, it suffices to prove that $\ker(\mathrm{sp}) = \ker(N)$ rationally.

Let $\Delta \to \Delta^k$, $u \mapsto (u, \ldots, u)$ be the diagonal cocharacter. Then the pullback of $X \to Y$ along Δ is a 1-parameter degeneration $X_\Delta \coloneqq X \times_Y \Delta \to \Delta$, whose singularities are analytically locally of the form

$$x_1^{(1)} \cdots x_1^{(n_1)} = \cdots = x_k^{(1)} \cdots x_k^{(n_k)} = u,$$

i.e. a fiber product of snc singularities. By subdividing into lattice simplices the corresponding dual polysimplex $\prod_{i=1}^k \sigma_{n_i - 1}$ to this stratum, we produce a toroidal resolution $X'_\Delta \to \Delta$ which is a semistable degeneration, and for which $\Gamma(X'_{\Delta,0}) \simeq \Gamma(X_0)$ is a subdivision of the dual complex.

In particular, we have a canonical isomorphism $H_1(\Gamma(X'_{\Delta,0}), \mathbb{Z}) \simeq H_1(\Gamma(X_0), \mathbb{Z})$ and furthermore, the specialization map sp_Δ for the 1-parameter semistable degeneration X'_Δ agrees with sp under this isomorphism. Thus, it suffices to prove that the kernel of $\mathrm{sp}_\Delta\colon H_1(X_t, \mathbb{Q}) \to H_1(\Gamma(X'_{\Delta,0}), \mathbb{Q})$ is $W_{-1} \otimes \mathbb{Q}$. The result now follows from [43, Sec. 1, p. 105]. □

Proposition 5.13. *Let $f\colon X \to Y$ be a D-nodal degeneration, with $0 \in Y$ and $t \in Y$ a nearby point. Let $T_i\colon H_1(X_t, \mathbb{Z}) \to H_1(X_t, \mathbb{Z})$ be the monodromy about a component $D_i \subset D$ passing through 0. Then*

$$N_i(x) = -\sum_{\{j,j'\}} (x \cdot \widetilde{\gamma}_{jj'}) \gamma_{jj'}$$

where the indices $\{j,j'\}$ run through all double loci $V_{jj'}$ over the general point of D_i, $\gamma_{jj'} \coloneqq [c_t^{-1}(p_{jj'})]$ and $\widetilde{\gamma}_{ij} \coloneqq [c_t^{-1}(V_{jj'})]$ for any point $p_{jj'} \in V_{jj'}$, where $c_t \colon X_t \to X_0$ is the continuous map from Proposition 5.10.

Proof. The formula follows from a theorem of Clemens which computes the monodromy of any semistable degeneration [20, Thm. 4.4]. Though the case at hand is easier, since we only have simple nodes in the general fiber over D_i and the computation is essentially the same as the Picard–Lefschetz formula. □

This formula is compatible with the formula r_iB_i for the monodromy bilinear form of a shifted matroidal degeneration $X(\underline{R},\mathscr{H}) \to \Delta^k$. Indeed, any double locus $V_{jj'}$ of the general fiber over the i-th coordinate hyperplane of Δ^k has, by construction, vanishing cycle $\gamma_{jj'} = \mathbf{x}_i \in \mathbf{N} \simeq \mathrm{gr}^W_{-2}H_1(X_t,\mathbb{Z})$. Thus, Proposition 5.13 gives

$$B_i(x,x) = \sum_{\{j,j'\}} (x \cdot \widetilde{\gamma}_{jj'})L(x,\mathbf{x}_i) \quad \text{for } x \in H_1(X_t,\mathbb{Z}),$$

where $\widetilde{\gamma}_{jj'} \in H_{2g-1}(X_t,\mathbb{Z})$ is defined as above and L is the principal polarization. But for all $\{j,j'\}$, we have $(- \cdot \widetilde{\gamma}_{jj'}) = L(-,\gamma_{jj'})$. Thus, $B_i(x,x) = r_i\mathbf{x}_i^2$ where r_i is the number of hyperplanes normal to $\mathbf{x}_i$ in the multiset $\mathscr{H}$.

5.3. **Resolution of the base change of a nodal morphism.** The goal of this section is to prove the following general theorem.

Theorem 5.14. *Let $\pi \colon Y' \to Y$ be a morphism, and let $f' \colon X' \to Y'$ be the base change of a strictly D-nodal morphism $f \colon X \to Y$ along π. Suppose furthermore that Y' is smooth and the reduction of $E \coloneqq \pi^{-1}(D)$ is an snc divisor.*

Then an ordering of the components of E, and an ordering of the components V_i over each D_i determines, in a canonical manner, a relatively projective resolution of singularities $X''' \to X'$ for which the morphism $f''' \colon X''' \to Y'$ is strictly E-semistable, and an intermediate partial resolution $X''' \to X'' \to X'$ for which $f'' \colon X'' \to Y'$ is strictly nearly E-nodal.

This theorem can be viewed as an explicit special case of the functoriality theorem for multivariable semistable reductions, see [2, Thm. 4.4]. The proof works by observing that the base change is locally toroidal. This allows us to apply toroidal resolutions locally, which glue to a global resolution.

Proof. Define a bijection between the components of E and the non-negative integers

$$\{1,\ldots,\#\text{ components of } E\}, \tag{29}$$

increasing in the total order, so that any snc stratum $E_J \coloneqq \bigcap_{j\in J} E_j$ defines a unique subset of (29). Suppose that E_J is an snc stratum of codimension n in Y', so that $|J| = n$. Say $J =$

$\{1, \ldots, n\}$ for indexing convenience. At any point in the open snc stratum E_J^o, the hypothesis that $E = \pi^{-1}(D)$ is snc implies that π induces a local monomial transform

$$
\begin{aligned}
u_1 &= w_1^{r_{11}} \cdots w_n^{r_{1n}} =: w^{\vec{r}_1}, \\
&\cdots \\
u_k &= w_1^{r_{k1}} \cdots w_n^{r_{kn}} =: w^{\vec{r}_k}.
\end{aligned} \tag{30}
$$

where u_i are a subset of local coordinates on Y, which cut out the stratum $D_I = \bigcap_{i \in I} D_i$ into which E_J maps, and w_j cuts out E_j. By the hypothesis that f is D-nodal, the base change f' has a local form which is the product of a smooth morphism with

$$\{x_1 y_1 = w^{\vec{r}_1}, \cdots, x_m y_m = w^{\vec{r}_m}\} \mapsto (w_1, \ldots, w_n), \tag{31}$$

up to relabeling the indices $\{1, \ldots, m\}$ of the fiber components.

Note that x_i and y_i are local equations of components of $V_i \subset X_i$ over D_i. By convention, take x_i to cut out the component earlier in the total order (here we use smoothness of V_i to ensure that $x_i = y_i = 0$ is not a self-nodal locus of a component).

We will first construct the partial resolution $f'' \colon X'' \to Y$ which is nearly E-nodal. The equations (31) define a morphism of toric varieties. The domain of (31) is described by the normal fan of a polytope $P(\vec{r}_1, \ldots, \vec{r}_n)$, which we define now.

Let $b_i \colon \mathbb{R} \to \mathbb{R}^J \simeq \mathbb{R}^n$ for $i = 1, \ldots, k$ be the $\mathbb{Z}$-piecewise linear function

$$b_i(z) := \begin{cases} -z\vec{r}_i & \text{if } z \leq 0, \\ 0 & \text{if } z \geq 0, \end{cases}$$

and let $b(z_1, \ldots, z_k) := \sum_{i=1}^k b_i(z_i)$. Then the graph $\Gamma(b) \subset \mathbb{R}^k \times \mathbb{R}^J$ is the boundary of the polytope $P(\vec{r}_1, \ldots, \vec{r}_n) := \Gamma(b) + (\mathbb{R}_{\geq 0})^J$—the monomials x_i and y_i respectively correspond to the primitive integral vectors along the restriction of the graph of b, to the positive- and negative i-th coordinate axis $\mathbb{R} \subset \mathbb{R}^k$, respectively.

The *bending parameter* of a piecewise linear function $b_0 \colon \mathbb{R} \to \mathbb{R}^J$ at $z = z_0$ is defined by $\frac{\partial b_0}{\partial z}(z_0 + \epsilon) - \frac{\partial b_0}{\partial z}(z_0 - \epsilon) \in \mathbb{R}^J$, for $\epsilon \ll 1$. Then the function $b_i \colon \mathbb{R} \to \mathbb{R}^J$ above is uniquely characterized by the following properties:

(1) $b_i(z) = 0$ for $z \gg 0$,
(2) b_i only bends at $z = 0$, and
(3) the bending parameter at $z = 0$ is $\vec{r}_i = r_{i1}e_1 + \cdots + r_{in}e_n \in \mathbb{R}^J$.

Fix a very large integer $N \gg 0$. We may uniquely define a (continuous, piecewise linear) function $c_i \colon \mathbb{R} \to \mathbb{R}^J$ for $i = 1, \ldots, m$ by the following properties:

(1) $c_i(z) = 0$ for $z \gg 0$,
(2) $c_i(z)$ only bends at $z = jN + \ell$ for $j \in J$ and $\ell \in \{1, \ldots, r_{ij}\}$ and
(3) the bending parameter at $z = jN + \ell$ is the basis vector $e_j \in \mathbb{R}^J$.

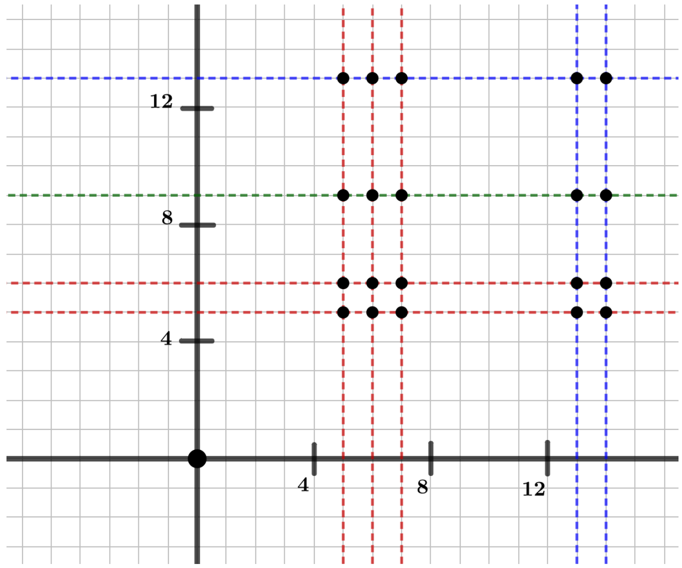


FIGURE 18. Domains of linearity of b and $c\colon \mathbb{R}^2 \to \mathbb{R}^3$. The relevant monomial transform is $u_1 = w_1^3 w_3^2$ and $u_2 = w_1^2 w_2 w_3$. The bending locus of b is depicted in solid black, while the bending locus of c is depicted in dotted red, green, and blue lines. The red, green, and blue lines have bending parameter e_1, e_2, and e_3, respectively. Here, we took $N = 4$.

See Figure 18 for a depiction of the domains of linearity in an example, when $k = 2$. We define $c\colon \mathbb{R}^k \to \mathbb{R}^J$ by the formula $c(z_1, \dots, z_k) := \sum_{i=1}^k c_i(z_i)$. Then since each c_i is convex, $\Gamma(c)$ is the boundary of a polytope $Q(\vec{r}_1, \dots, \vec{r}_n) := \Gamma(c) + (\mathbb{R}_{\geq 0})^J$.

Note that the normal fan $\mathfrak{F}_J$ of $Q(\vec{r}_1, \dots, \vec{r}_n)$ depends only on labelling of the indices $J = \{1, \dots, n\}$. Furthermore, the normal fan of $Q(\vec{r}_1, \dots, \vec{r}_n)$ is a refinement of the normal fan of $P(\vec{r}_1, \dots, \vec{r}_n)$ since the linear parts of $b_i(z)$ and $c_i(z)$ are the same for any $z < 0$ or $z \gg 0$, e.g. $z > (\#\text{ components of } E) \cdot N + \max r_{ij}$ suffices.

Thus, in a neighborhood of a point $p \in (f')^{-1}(E_J)$, the normal fan of $Q(\vec{r}_1, \dots, \vec{r}_n)$ defines a toroidal birational morphism $X''_p \to X'_p$. To check that this birational modification is globally well-defined, it suffices to prove that these birational modifications are compatible with the incidences $E_J \subset E_{J'}$ for some $J' \subset J$, and that they are compatible on overlapping charts over a given stratum E_J. The latter is automatic since the modification $X''_p \to X'_p$ depended only on the ordering of J and the snc divisor E has global normal crossings.

To check the compatibility between strata, the restriction of the morphism of fans

$$\mathfrak{F}_J \to \prod_{j \in J} \mathbb{R}_{\geq 0} e_j^\vee$$

to the coordinate subspace $\mathfrak{F}_{J'} \to \prod_{j=J'} \mathbb{R}_{\geq 0} e_j^\vee$ should be the normal fan of the corresponding polytope for J'. Dually, in terms of the above defined PL function $c = c^J : \mathbb{R}^k \to \mathbb{R}^J$, if we consider the projection $p_{J,J'} : \mathbb{R}^J \to \mathbb{R}^{J'}$, then the composition

$$\mathbb{R}^k \xrightarrow{c_J} \mathbb{R}^J \xrightarrow{p_{J,J'}} \mathbb{R}^{J'}$$

should agree with the PL function $c_{J'}$ and indeed it does—this projection simply forgets the bending (3) along any $z_i = jN + \ell$ for the $j \in J \setminus J'$ and any $i \in \{1, \ldots, m\}$. In Figure 18, this corresponds to forgetting the colors indexed by $J \setminus J'$.

Hence, there is a globally well-defined toroidal birational modification $X'' \to X'$ which is locally defined by the morphism from the normal fan of $Q(\vec{r}_1, \ldots, \vec{r}_n)$ to the normal fan of $P(\vec{r}_1, \ldots, \vec{r}_n)$. It is furthermore relatively projective, since we defined it in terms of polytopes.

We claim that $f'' : X'' \to Y'$ has nearly E-nodal singularities. We check that $f'' : X'' \to Y'$ has the desired local form of Definition 5.1(2) by examining a neighborhood of a face of the polytope $Q(\vec{r}_1, \ldots, \vec{r}_n)$, i.e. a neighborhood of a domain of linearity of the function $c : \mathbb{R}^k \to \mathbb{R}^J$. Such a domain of linearity is given by equations

$$\textstyle\bigcap_{i \in I} \{z_i = j_i N + \ell_i\}$$

for some subset $I \subset \{1, \ldots, k\}$ and some indices $j_i \in J$. In the neighborhood of such a domain, the bending parameter of $c_i(z_i)$ is e_{j_i} and thus, the local equation of the morphism f'' is

$$x_1 y_1 = w_{j_1}, \quad \cdots \quad, x_m y_m = w_{j_m}$$

where x_i and y_i are the local equations of the reduced union of components corresponding, respectively, to the facets of $Q(\vec{r}_1, \ldots, \vec{r}_n)$ given by $z_i \leq j_i N + \ell_i$ and $z_i \geq j_i N + \ell_i$. Thus, f'' has nearly E-nodal singularities, which are E-nodal if and only if the j_i are distinct, ranging over all possible strata over all E_J.

Thus, we have completed our first goal: producing a birational modification $X'' \to X'$ for which $f'' : X'' \to Y$ is nearly E-nodal. But, as noted before, X'' may not be smooth, due to the presence of the local form

$$x_1 y_1 = \cdots = x_m y_m = w, \tag{32}$$

and products thereof. The fan of this local form may be described as follows: Let $[0,1]^m \subset \mathbb{R}^m$ denote the unit cube and let $\operatorname{Cone}[0,1]^m \subset \mathbb{R}^m \times \mathbb{R}$ denote the cone over the cube. Then, the morphism to $\mathbb{C}_w$ is given by the morphism of fans $\operatorname{Cone}[0,1]^m \to \mathbb{R}_{\geq 0}$ which is projection to the last coordinate.

We may define a small, regular resolution of (32) by subdividing the fan into standard affine cones, in a manner which introduces no new rays. Note that the original rays, corresponding to the components over $w = 0$, are the cones over the 2^m vertices of the cube. Equivalently, we must decompose the cube $[0,1]^m$ into lattice simplices of minimal volume $(1/m!)$. Such a subdivision arises from a sequence of toric blow-ups, by blowing up the components of the fiber over $w = 0$ of (32) in any order. Blowing up the component corresponding of a vertex of the cube $[0,1]^m$ produces a subdivision of the cube which inserts all diagonals of the cube emanating from that vertex. The resulting polyhedral cells are cones over lower dimensional cubes. Further blow-ups further subdivide these cones, until all cells are standard simplices.

Thus, to define globally a projective subdivision, requires a total ordering of the components of $f'' \colon X'' \to Y$ over each component $E_j \subset E$. Since (32) only involves a single coordinate u, it suffices to resolve by blowing up, in the specified order, the components over each $E_j = V(w_j)$. Then, over a deeper stratum E_J, these blow-ups are a product of blow-ups of the local forms (32), ranging over $j \in J$. Thus, they induce the product resolution over deeper strata E_J.

The total ordering of all components V_i over D_i for all $i \in I$ induces a total order on the components over E_j. For instance, the total order on the original components induces a lexicographical ordering on the components introduced by the partial resolution $X'' \to Y'$ over each E_j, which are naturally indexed by the top-dimensional cells of a cuboid whose corners are the strict transforms of components of the fibers of $X' \to Y'$ over E_j, see Figure 18. Blowing up these components in order, we produce a subdivision for each (singular) stratum over E_j, giving a projective resolution of singularities $X''' \to X''$.

Examining the cones of the resulting fan for X''', we see that the local form for $f''' \colon X''' \to Y'$ is given by a product of morphisms of fans of the form

$$\prod [\mathrm{Cone}\{0 \le z_1 \le \cdots \le z_m \le 1\} \to \mathbb{R}_{\ge 0}]$$

(here, we allow $m = 1$ to include factors which are smoothings of nodes) with a smooth morphism. We deduce that the morphism f''' is E-semistable. $\square$

Remark 5.15. Suppose $\pi \colon Y' \to Y$ is a birational modification. Consider the strict transforms $E_i \coloneqq \pi_*^{-1} D_i$. For any stratum $E_J \subset E_i$ contained in the strict transform (i.e. $J \ni i$), the local monomial transform (30) is of the form $u_j = w^{\vec{r}_j}$ where $w^{\vec{r}_j}$ does not involve the variable w_i for all $j \neq i$, and $w^{\vec{r}_i} = w_i \cdot$(a monomial in w_j for $j \neq i$). Thus, in Theorem 5.14, nodes over D_i are in natural bijection with the nodes over E_i and indeed, $E_i^o \to \pi(E_i^o) \subset D_i^o$ is an isomorphism onto its image, with the restriction of the map $X''' \to X$ an isomorphism. So, at least over E_i^o, the morphism $f''' \colon X''' \to Y'$ is E-nodal.

More generally, when π is an alteration, the nodes over E_i^o are étale over the nodes of D_i^o.

5.4. **Resolution of the base change of a transversely shifted matroidal degeneration.** Suppose that $X = X(\underline{R}, \mathscr{H})$ is a transversely shifted matroidal degeneration, so that, in particular, X is regular and $f \colon X \to \Delta^k$ has D-nodal singularities (Prop. 5.3). Then f is strictly D-nodal if and only if for each element $\mathbf{x}_i \in \mathbf{N}$ of the matroid $\underline{R}$, there are at least two hyperplanes with normal vector $\mathbf{x}_i$ (Prop. 5.5). In this case, we may directly apply the resolution algorithm of Theorem 5.14. But in fact, even if an irreducible component over $V(u_i)$ is self-nodal, the two branches are not permuted by monodromy, because it is possible to choose globally a normal vector to a hyperplane $H \in \mathscr{H}$. So the resolution algorithm of Theorem 5.14

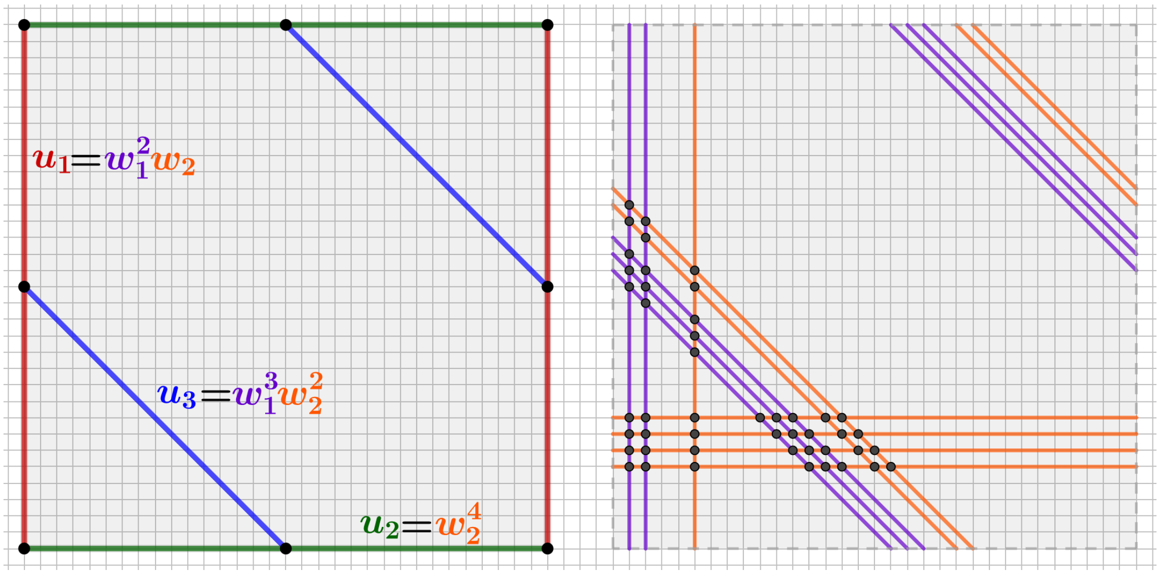


FIGURE 19. Left: Bending of b_1, b_2, b_3 for $X(32 \mid b_1, b_2, b_3)$ a shifted matroidal degeneration. Right: Bending of c_1, c_2 for the nearly E-nodal partial resolution of the base change, in purple, orange, respectively. Grid points are $(\frac{1}{32}\mathbb{Z})^2$. We have taken $N = 4$, as in Figure 18.

still works analytically, by consistently choosing one of the two branches and performing the blow-up in local charts, as in the proof. But, since one does not blow up global Weil divisors, it is unclear whether the result is, in general, projective.

Example 5.16. Let $X(\underline{R}, \mathscr{H}) = X(2 \mid b_1, b_2, b_3)$ be the transversely shifted matroidal degeneration depicted in the righthand side of Figure 14. Pass to a Veronese embedding for some large $d \gg 0$ (we take 32 times the principal polarization, in the present example).

Consider the monomial base change $\Delta^2 \to \Delta^3$ given by

$$u_1 = w_1^2 w_2, \quad u_2 = w_2^4, \quad u_3 = w_1^3 w_2.$$

The pullback and its nearly E-nodal resolution $X(32 \mid c_1, c_2)$ are depicted in Figure 19, where $E \coloneqq V(w_1 w_2)$ is the reduced inverse image of $D \coloneqq V(u_1 u_2 u_3)$.

The original red hyperplane in the left of Figure 19 generates hyperplanes to its right, in the direction of positive intersections with $\mathbf{x}_1 = (1, 0)$, the green hyperplane generates hyperplanes above it, in the direction of positive intersections with $\mathbf{x}_2 = (0, 1)$, and the blue hyperplane generates hyperplanes above and to the left, in the direction of positive intersection with $\mathbf{x}_3 = (1, 1)$. Purple hyperplanes (corresponding to u_1), always precede orange hyperplanes (corresponding to u_2), because of the ordering on the components $V(u_1)$, $V(u_2)$ of E.

The righthand figure is then the Mumford construction which describes the nearly E-nodal partial resolution, as in Theorem 5.14, of the base-changed Mumford construction.

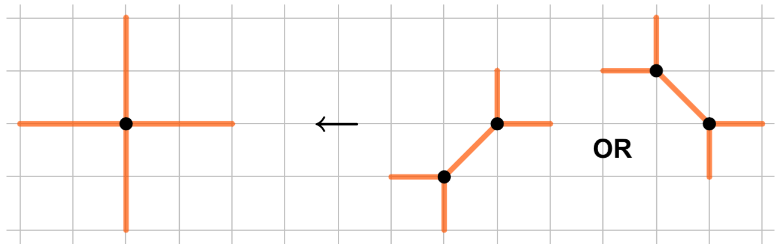


FIGURE 20. The two possible small resolutions for a 4-valent intersection point of 2-dimensional cuboids in $\mathrm{Bend}(c_i)$. The normal fan is a subdivision of $[0,1]^2$.

In terms of Mumford constructions, the further small resolution to an E-semistable morphism is a bit more complicated to describe. Essentially, the relevant blow-ups resolve the 4-valent intersection points of the (monochromatic) cuboids in $\mathrm{Bend}(c_i)$ for $i = 1, 2$, into two 3-valent intersection points, in a manner which is locally of the form shown in Figure 20.

6. The second Voronoi fan and Alexeev's theorem

6.1. **The universal family of abelian torsors with theta divisor.** One of the most celebrated applications of the Mumford construction is the modular compactification of the moduli space $\mathcal{A}_g$ of PPAVs of dimension g, due to Alexeev [4], building on work of Namikawa, Nakamura, and Faltings–Chai [47, 48, 49, 50, 17, 7]; see [51, Thm. 9.20].

In previous sections, we have extracted from a section $\overline{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ on a torus $\mathbb{T}^g = \mathbf{M}_\mathbb{R}/\mathbf{M}$, or its PL lift $b_i \colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$, an integral bilinear form $B_i \in \mathrm{Sym}^2\mathbf{M}^\vee$. Here we reverse this procedure, extracting from a bilinear form B_i a PL function b_i with periodic bending locus. In this manner, we produce both a canonical choice of fan for $\mathcal{A}_g$ (see Def. 2.24), and a "tautological" Mumford construction over its cones. The procedure is straightforward: we graph (a function closely related to) $B_i(\mathbf{m}, \mathbf{m})$ over the lattice points $\mathbf{m} \in \mathbf{M}$, take the convex hull of the corresponding integral points, and take the unique PL function whose graph is the boundary of this hull.

Definition 6.1. Let $B \in \mathcal{P}_g$ be a positive-definite symmetric bilinear form on $\mathbf{M}_\mathbb{R}$. It defines a square-distance function d_B on $\mathbf{M}_\mathbb{R}$ by $x \mapsto B(x, x)$. The *Voronoi decomposition* Vor_B of $\mathbf{M}_\mathbb{R}$ is the one whose maximal open polyhedral cells are defined as follows:

$$\mathrm{Vor}_{B,\mathbf{m}} = \{x \in \mathbf{M}_\mathbb{R} \,|\, d_B(x, \mathbf{m}) < d_B(x, \mathbf{m}') \text{ for all } \mathbf{m}' \in \mathbf{M} \setminus \mathbf{m}\},$$

ranging over all $\mathbf{m} \in \mathbf{M}$. That is, the maximal cells are those points closer (with respect to d_B) to one lattice point $\mathbf{m} \in \mathbf{M}$ than any other.

The *Delaunay decomposition* Del_B is the polyhedral decomposition of $\mathbf{M}$ whose cells are dual to the cells of the Voronoi decomposition, and whose vertices are $\mathbf{M} \subset \mathbf{M}_\mathbb{R}$.

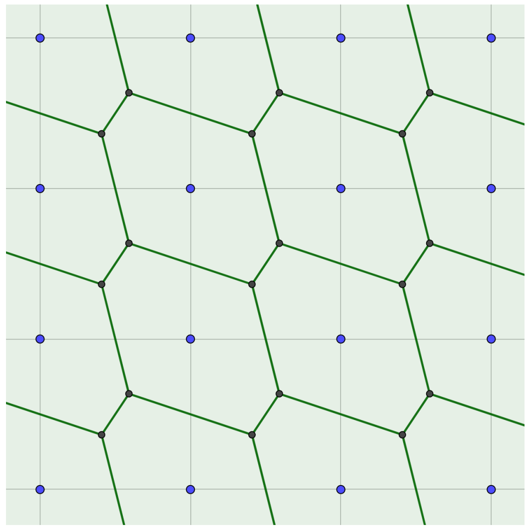


FIGURE 21. Voronoi cells for $4\mathbf{x}^2 + 2\mathbf{x}\mathbf{y} + 3\mathbf{y}^2$. Lattice points $m \in \mathbf{M}$ in blue.

Example 6.2. Consider the bilinear form B on $\mathbf{M}_\mathbb{R} \simeq \mathbb{R}^2$ corresponding to the matrix

$$B = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$$

from Figure 8 and Example 3.34. The associated square distance function is given by the quadratic form $4\mathbf{x}^2+2\mathbf{x}\mathbf{y}+3\mathbf{y}^2$. The Voronoi cells are depicted in Figure 21. The corresponding Delaunay decomposition is depicted in Figure 7.

Remark 6.3. When B is degenerate, the Voronoi cells are still defined, but they are of infinite volume, as they are invariant under translation by the null subspace of B.

On the one hand, the Voronoi decomposition varies continuously with $B \in \mathcal{P}_g^+$, and while its cells are polytopes, they are not integral. On the other hand, the Delaunay decomposition has integral polytope cells, which do not vary continuously, but rather are constant along the relative interiors of the cones of a fan:

Definition 6.4. The *second Voronoi fan* $\mathfrak{F}_{\rm vor}$ is the polyhedral decomposition of $\mathcal{P}_g^+$ whose cones are the closures of loci on which the Voronoi decomposition is combinatorially constant, or equivalently, on which the Delaunay decomposition is constant. More precisely, $B, B' \in \mathcal{P}_g^+$ are in the relative interior τ^o of the same cone $\tau \in \mathfrak{F}_{\rm vor}$ if and only if B and B' are connected by a path along which the Delaunay decomposition is constant. The *second Voronoi compactification* is the toroidal compactification (see Section 2.5)

$$\mathcal{A}_g \hookrightarrow \overline{\mathcal{A}}_g^{\rm vor} := \overline{\mathcal{A}}_g^{\mathfrak{F}_{\rm vor}}.$$

As required by Definition 2.24, $\mathfrak{F}_{\mathrm{vor}}$ is invariant under the action of $A \in \mathrm{GL}_g(\mathbb{Z})$ on $B \in \mathcal{P}_g^+$ via $B \mapsto ABA^T$, as these transformations correspond to changes-of-basis of the lattice $\mathbf{M} \simeq \mathbb{Z}^g$. It is a theorem due to Voronoi that the number of $\mathrm{GL}_g(\mathbb{Z})$-orbits of cones is finite.

Construction 6.5 (Mumford construction of second Voronoi type)**.** Over each cone $\mathbb{B} \in \mathfrak{F}_{\mathrm{vor}}$ intersecting $\mathcal{P}_g^+$ there is a "tautological" Mumford construction, which we will now define.

Let $B_i \in \mathbb{B}$ be primitive integral vectors generating the rays of $\mathbb{B}$. Then, by considering the characteristic vector of the bilinear form B_i mod 2, it is possible to choose *characteristic* linear forms $L_i \colon \mathbf{M} \to \mathbb{Z}$ for which

$$\mathbf{m} \mapsto b_i(\mathbf{m}) \coloneqq \frac{B_i(\mathbf{m}, \mathbf{m}) - L_i(\mathbf{m})}{2} \tag{33}$$

is integer-valued on $\mathbf{M} \simeq \mathbb{Z}^g$. For instance, we may take the coefficients of L_i to be the diagonal entries of the matrix B_i in some basis. Then, there is a unique convex section $\overline{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ admitting a lift to $\mathbf{M}_{\mathbb{R}}$ which agrees with the above function b_i on $\mathbf{M}$.

It is a simple verification from Definition 6.1 that $\bigcup_i \mathrm{Bend}(\overline{b}_i)$ is exactly $\mathrm{Del}_{\sum r_i B_i}$ for any $(r_1, \ldots, r_k) \in \mathbb{N}^k$—this condition translates into a condition that the bending locus of the convex $\mathbb{Z}\mathrm{PL}$ function which agrees with $\mathbf{m} \mapsto \sum_{i=1}^k r_i b_i(\mathbf{m})$ on $\mathbf{M}$ is the same for all $(r_1, \ldots, r_k) \in \mathbb{N}^k$. Thus, the $\overline{b}_i$ are dicing. Furthermore, the additional condition of Construction 3.26 is satisfied: The associated bilinear forms B_i span extremal rays of a polyhedral cone in $(\mathcal{A}_g)_{\mathrm{trop}}$. Thus, we get a relatively proper extension of the universal family $X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$. ♣

Construction 6.6. We now construct a torsor $\mathcal{X}_g^\star \to \mathcal{A}_g$ over the universal abelian variety $\mathcal{X}_g \to \mathcal{A}_g$ and an extension of it over the second Voronoi compactification.

The issue begins in the interior $\mathcal{A}_g$, see e.g. the discussion in [32, Sec. 1] and [49, Sec. 19 and bottom of p. 209]: For a given abelian variety $(A, 0, L)$ with origin $0 \in A$ and principal polarization $L \in \mathrm{NS}(A)$, there are 2^{2g} different possible (-1)-symmetric lifts $\mathcal{L} \in \mathrm{Pic}(A)$ of L. These lifts define naturally a torsor over the 2-torsion subgroup $A[2]$ and thus, on the universal family $\mathcal{X}_g \to \mathcal{A}_g$ we have a natural torsor $\mathrm{Lifts}(L) \to \mathcal{A}_g$ under the group scheme $\mathcal{X}_g[2] \to \mathcal{A}_g$ of relative 2-torsion in $\mathcal{X}_g \to \mathcal{A}_g$. But $\mathrm{Lifts}(L)$ admits no section—it is impossible to globally lift L to some (-1)-symmetric $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_g/\mathcal{A}_g)$ when $g \geq 2$.

There are two ways to resolve the issue: Either one passes to a finite étale cover $\widetilde{\mathcal{A}}_g \to \mathcal{A}_g$ over which this torsor is trivialized, or one defines a new universal family $\mathcal{X}_g^\star \to \mathcal{A}_g$ of abelian torsors $(X, \mathcal{L})$, with a lift of the principal polarization to a line bundle.

The family $\mathcal{X}_g^\star$ will be, étale-locally over $\mathcal{A}_g$, isomorphic to $\mathcal{X}_g \to \mathcal{A}_g$. Over an étale open chart $U_i \to \mathcal{A}_g$ over which there is a lift $\mathcal{L}$ of L, we have a family $((\mathcal{X}_g)_{U_i}, \mathcal{L}_{U_i}) \to U_i$. We may uniquely glue these families over the double overlaps $U_i \cap U_j$ to produce a universal family $(\mathcal{X}_g^\star, \mathcal{L}) \to \mathcal{A}_g$. Notably, the gluing of $(\mathcal{X}_g)_{U_i}$ and $(\mathcal{X}_g)_{U_j}$ may not respect the origin sections, but must respect the lift $\mathcal{L}$.

For a cone $\mathbb{B} \in \mathfrak{F}_{\mathrm{vor}}$, Construction 6.5 gives a Mumford construction $X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$. In the category of DM analytic stacks, this family descends *as a family of polarized varieties* over an étale neighborhood of the boundary strata of $\mathcal{A}_g^{\mathbb{B}}$. The reason is that $\overline{b}_i \in H^0(\mathbb{T}^g, \mathbb{Z}\mathrm{PL}/\mathbb{Z}\mathrm{L})$ which define the relevant polytopal Mumford Construction 3.26 are defined canonically by the B_i—one may worry that some non-canonicity is introduced by the choice of the characteristic linear form $L_i(\mathbf{m})$ in (33) which determine the lifts b_i. But a different choice $L_i \mapsto L_i + 2L_i'$ produces the same section $\overline{b}_i$.

On the other hand, due to the shifts $L_i(\mathbf{m})$, the resulting Mumford construction has no canonical origin section, see Remark 3.39. Thus, the output of Construction 6.5 does not glue canonically (i.e. in a manner independent of the choice of L_i) to the universal family $\mathcal{X}_g \to \mathcal{A}_g$ (which has an origin section), but rather to the universal family $\mathcal{X}_g^\star \to \mathcal{A}_g$ (which has a canonical lift of the principal polarization). If one were to take $L_i = 0$ in (33), we would retain a canonical origin point, but the lift b_i fails to have integral slopes, leading to non-reduced fibers, see [49].

By their canonicity and the uniqueness of gluings, the Mumford constructions of Construction 6.5 are compatible between adjacencies of cones in $\mathfrak{F}_{\mathrm{vor}}$. Thus, we may glue them via the unique gluings respecting the lift of L, to produce a proper extension

$$\overline{\mathcal{X}_g^\star}^{\mathrm{vor}} := \mathcal{X}_g^\star \cup \bigcup_{\mathbb{B} \in \mathrm{GL}_g(\mathbb{Z})\backslash \mathfrak{F}_{\mathrm{vor}}} X^{\mathrm{univ}}(\mathbb{b}) \to \overline{\mathcal{A}}_g^{\mathrm{vor}}.$$

In summary, $\overline{\mathcal{X}_g^\star}^{\mathrm{vor}}$ admits a relatively projective, surjective morphism (a priori, just in the category of DM analytic stacks)

$$f_{\mathrm{vor}} \colon \overline{\mathcal{X}_g^\star}^{\mathrm{vor}} \to \overline{\mathcal{A}}_g^{\mathrm{vor}},$$

extending the universal family of abelian torsors with lift of principal polarization. It follows from Serre's GAGA for Deligne–Mumford stacks, see [62, Cor. 5.13], that $\overline{\mathcal{X}_g^\star}^{\mathrm{vor}}$ is a DM algebraic stack and f_{vor} is projective. For instance, f_{vor} becomes a morphism of projective schemes after taking the pullback toroidal compactification of an appropriate étale cover.

Our construction also produces an extension

$$f_{\mathrm{vor}} \colon (\overline{\mathcal{X}_g^\star}^{\mathrm{vor}}, \overline{\Theta}_g^{\mathrm{vor}}) \to \overline{\mathcal{A}}_g^{\mathrm{vor}}$$

of the universal pair (X, Θ). Here, the theta divisor $\Theta \in |\mathcal{L}|$ is the unique element of the linear system. Note that Θ extends as an effective, relatively ample divisor over $\overline{\mathcal{A}}_g^{\mathrm{vor}}$ as the vanishing locus of the unique weight $w = 1$ theta function $\Theta = V(\Theta_{(0/1,\dots,0/1)})$ of Construction 3.21. ♣

Example 6.7 (Second Voronoi fan for $g \leq 6$)**.** We now describe, in varying levels of detail, the second Voronoi fan of $\mathcal{A}_g$ for small dimensions, and the extension of the universal family over it, defined by Construction 6.6. This line of Russian mathematical inquiry is notable for extending across more than a century.

$\boxed{g = 1:}$ Here, any fan for $\mathcal{A}_1$ is the same, and there is only one Voronoi cone, corresponding to the ray $\mathbb{R}_{\geq 0}\{x^2\} \subset \mathcal{P}_1^+ \simeq \mathbb{R}_{\geq 0}$. The corresponding Delaunay decomposition is Figure 4, and the resulting Mumford construction is the Tate curve. The universal family

$$\overline{\mathcal{X}_1^\star}^{\mathrm{vor}} \to \overline{\mathrm{Sp}_2(\mathbb{Z})\backslash \mathcal{H}_1}^{\mathrm{vor}} \simeq \mathbb{P}(4, 6)$$

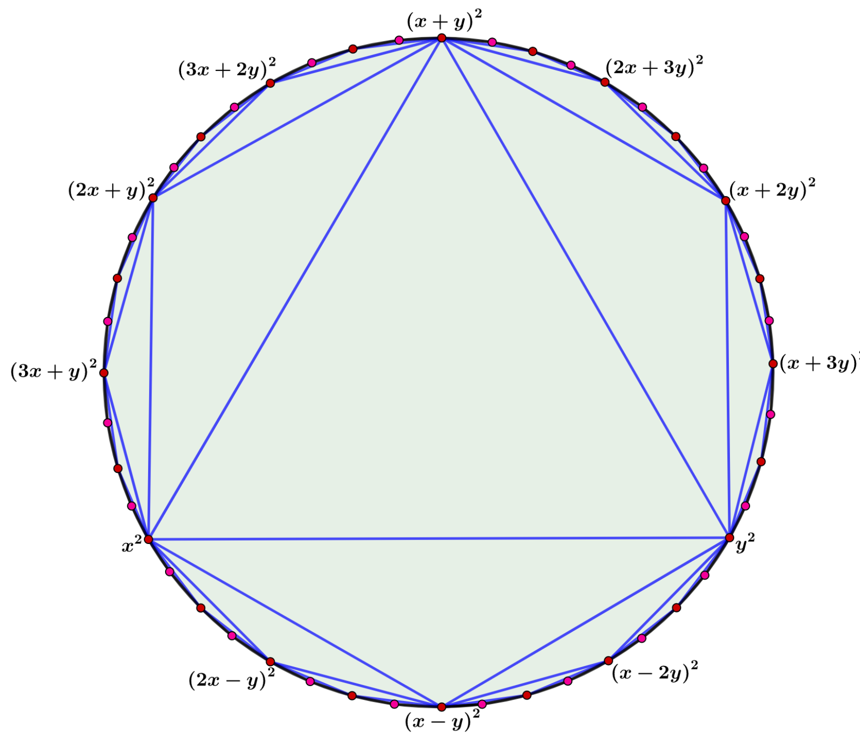


FIGURE 22. Projectivization of the second Voronoi fan decomposition of $\mathcal{P}_2^+$. Cones of dimension 3, 2, 1 in green, blue, red, respectively.

is the extension of the universal elliptic curve by a nodal elliptic curve. Over the coarse space $\mathbb{P}^1_j$ of $\mathbb{P}(4,6)$, i.e. the j-line, the nodal curve fibers over $j = \infty$.

$\boxed{g = 2:}$ Here, the second Voronoi fan is $\mathfrak{F}_{\rm vor} = \mathrm{GL}_2(\mathbb{Z}) \cdot \mathbb{R}_{\geq 0}\{\mathbf{x}^2, \mathbf{y}^2, (\mathbf{x}+\mathbf{y})^2\}$. See Figure 22. Thus, $\mathfrak{F}_{\rm vor}$ is the orbit of a single cographic cone, associated to the theta graph, with the lower dimensional faces corresponding to contractions of the theta graph (caveat lector: edge contractions of G give, in the sense of matroids, deletions of the cographic matroid $M^*(G)$).

There is one orbit each of 3-, 2-, 1-, and 0-dimensional cones, corresponding respectively to the cographic cones of the theta graph, the wedge of two circles, a single circle, and a point. Over a maximal, 3-dimensional cone, the universal family $\overline{\mathcal{X}_2^\star}^{\rm vor} \to \overline{\mathcal{A}}_2^{\rm vor}$ is extended by a Mumford construction $X(\mathbb{b}) \to \Delta^3$ isomorphic to Example 3.35. The reduction theory, i.e. analysis of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes, of positive-definite bilinear forms of rank 2, goes back at least to work of Fricke–Klein [37]; see Vallentin [64, Ch. 2] for some historical discussion.

$\boxed{g = 3:}$ Here, the second Voronoi fan is $\mathfrak{F}_{\rm vor} = \mathrm{GL}_3(\mathbb{Z}) \cdot \mathbb{R}_{\geq 0}\{\mathbf{x}_i^2, (\mathbf{x}_i - \mathbf{x}_j)^2\}$ for $1 \leq i < j \leq 3$. There is an analogous Voronoi cone in any rank g, called *Voronoi's principal domain of the first type*. It is the graphic cone $\mathbb{B}_{M(K_{g+1})}$ associated to the graphic matroid $M(K_{g+1})$ of the complete graph K_{g+1} on $g+1$ vertices. The number of $\mathrm{GL}_3(\mathbb{Z})$-orbits of cones of dimensions 6, 5, 4, 3, 2, 1, 0 are, respectively, 1, 1, 2, 2, 1, 1, 1. Over the maximal, 6-dimensional cone, the universal family $\overline{\mathcal{X}_3^\star}^{\mathrm{vor}} \to \overline{\mathcal{A}}_3^{\mathrm{vor}}$ is extended by a Mumford construction $X(\mathbb{b}) \to \Delta^6$ associated to the cographic or graphic matroid of the complete graph K_4 (note that $M(K_4) \simeq M^*(K_4)$ since K_4 is a planar, self-dual graph). See Example 4.13.

$\boxed{g = 4:}$ Here, the second Voronoi fan is $\mathfrak{F}_{\mathrm{vor}} = \mathrm{GL}_4(\mathbb{Z}) \cdot \{\mathbb{B}_{\mathrm{black}}, \mathbb{B}_{\mathrm{grey}}, \mathbb{B}_{\mathrm{white}}\}$, see for instance [64, Sec. 4.4.1]. That is, there are three $\mathrm{GL}_4(\mathbb{Z})$-orbits of maximal, 10-dimensional cones of $\mathfrak{F}_{\mathrm{vor}}$. The original computation goes back to Delaunay [24, Thm. III], who found all but one of the $\mathrm{GL}_4(\mathbb{Z})$-orbits of cones of $\mathfrak{F}_{\mathrm{vor}}$, and Shtogrin [59], who filled the gap.

The cone $\mathbb{B}_{\mathrm{black}}$ is Voronoi's principal domain of the first type, but unlike for $g \leq 3$, it no longer forms a fundamental domain for the action of $\mathrm{GL}_4(\mathbb{Z})$. It is a matroidal cone, associated to the graphic matroid $M(K_5)$ of the complete graph K_5. This cone is *not* cographic—the dual of a graphic matroid is graphic if and only if the graph is planar, and K_5 is not planar.

The cones $\mathbb{B}_{\mathrm{grey}}$ and $\mathbb{B}_{\mathrm{white}}$ are simplicial, but are not matroidal—they both have one ray generated by the positive-definite quadratic form giving the D_4-lattice, whereas all rays of a matroidal cone are quadratic forms of rank 1. There is one additional maximal, matroidal cone $\mathbb{B}_{M^*(K_{3,3})}$ of dimension 9. It is the matroidal cone of the cographic matroid $M^*(K_{3,3})$ of the complete bipartite graph $K_{3,3}$ and is the facet shared between two white cones.

The number of orbits of cones of dimensions 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 are, respectively, 3, 4, 7, 11, 11, 9, 7, 4, 2, 2, 1, see Hulek–Tommasi [35, p. 232]. In particular, the toroidal compactification $\overline{\mathcal{A}}_4^{\mathrm{vor}}$ has 2 boundary divisors and 3 zero-dimensional strata.

$\boxed{g = 5:}$ Correcting the nearly complete computations of Baranovskii–Ryshkov [56] to find one missing case, Peter Engel [26] verified by computer that there are 222 maximal, 15-dimensional Voronoi cones for $g = 5$. There are 9 rays, giving the boundary divisors of $\overline{\mathcal{A}}_5^{\mathrm{vor}}$. Dutour Sikirić et al. [25] proved that there are 110305 total $\mathrm{GL}_5(\mathbb{Z})$-orbits of cones in $\mathfrak{F}_{\mathrm{vor}}$ (*loc. cit.* give a slightly smaller number, as they only count cones which intersect $\mathcal{P}_5$).

There are 4 maximal, matroidal cones, of dimensions 15, 12, 12, 10. The first of these is Voronoi's principal domain of the first type $\mathbb{B}_{M(K_6)}$ and the last of these is the matroidal cone $\mathbb{B}_{\underline{R}_{10}}$ associated to the Seymour–Bixby matroid, see Example 4.7. The two maximal, matroidal cones of dimension 12 are the cographic cones of two trivalent genus 5 graphs (one of which is the 1-skeleton of a cube).

$\boxed{g = 6:}$ By work of Danilov–Grishukhin [23, Sec. 9], there are 11 maximal matroidal cones, with 8 cographic of dimension 15, and the remaining three of dimensions 21, 16, 12. Respectively, these are the graphic cone of K_7 and two matroidal cones, associated to regular matroids on 16 and 12 elements which are neither graphic nor cographic. The number of orbits of maximal cones is unknown, but exceeds $567,613,632$ by computations of Baburin–Engel [10].

Remark 6.8. Construction 6.6 shows that there exists a canonical element of the group $H^1(\mathcal{A}_g, \mathcal{X}_g[2])$ giving the abelian torsor $\mathcal{X}_g^\star$. We have an isomorphism with the group cohomology $H^1(\mathcal{A}_g, \mathcal{X}_g[2]) \simeq H^1(\mathrm{Sp}_{2g}(\mathbb{Z}), (\mathbb{Z}/2\mathbb{Z})^{2g})$ where $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $(\mathbb{Z}/2\mathbb{Z})^{2g}$ by the standard representation. Furthermore, the class of $\mathcal{X}_g^\star$ is nontrivial for $g \geq 2$, cf. [32]. It is natural to ask whether for $g \geq 2$ the (étale) Tate–Shafarevich group $H^1(\mathcal{A}_g, \mathcal{X}_g)$ satisfies $H^1(\mathcal{A}_g, \mathcal{X}_g) \simeq \mathbb{Z}/2\mathbb{Z}$, where we view $\mathcal{X}_g$ as a group scheme over $\mathcal{A}_g$. An affirmative answer would show that $\mathcal{X}_g^\star$ is the only abelian torsor under the universal abelian variety for $g \geq 2$.

The question of whether $H^1(\mathcal{A}_g, \mathcal{X}_g) \simeq \mathbb{Z}/2\mathbb{Z}$ is equivalent to the question of whether we have $H^1(\mathrm{Sp}_{2g}(\mathbb{Z}), (\mathbb{Z}/2\mathbb{Z})^{2g}) \simeq \mathbb{Z}/2\mathbb{Z}$, as the map

$$H^1(\mathrm{Sp}_{2g}(\mathbb{Z}), (\mathbb{Z}/2\mathbb{Z})^{2g}) \simeq H^1(\mathcal{A}_g, \mathcal{X}_g[2]) \to H^1(\mathcal{A}_g, \mathcal{X}_g)$$

is an isomorphism. To see this, note first that by [55, Prop. XIII.2.3], the group $H^1(\mathcal{A}_g, \mathcal{X}_g)$ is torsion. Second, for each $n \in \mathbb{N}$, the natural map $H^1(\mathcal{A}_g, \mathcal{X}_g[n]) \to H^1(\mathcal{A}_g, \mathcal{X}_g)[n]$ is an isomorphism by the long exact sequence in cohomology arising from the short exact sequence

$$0 \to \mathcal{X}_g[n] \to \mathcal{X}_g \xrightarrow{\cdot n} \mathcal{X}_g \to 0;$$

moreover, the group $H^1(\mathcal{A}_g, \mathcal{X}_g) = H^1(\mathrm{Sp}_{2g}(\mathbb{Z}), \mathbb{Z}^{2g})$ is 2-torsion, because $\mathrm{Sp}_{2g}(\mathbb{Z})$ contains an element that acts as -1 on $\mathbb{Z}^{2g}$.

A computation via the description $\mathrm{SL}_2(\mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z}) *_{(\mathbb{Z}/2\mathbb{Z})} (\mathbb{Z}/6\mathbb{Z})$ shows that $H^1(\mathcal{A}_1, \mathcal{X}_1) \simeq H^1(\mathrm{SL}_2(\mathbb{Z}), (\mathbb{Z}/2\mathbb{Z})^2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, even though $\mathcal{X}_1^\star \simeq \mathcal{X}_1$. Thus, there exists a universal non-trivial torsor under $\mathcal{X}_1 \to \mathcal{A}_1$, i.e. a family of genus one curves over $\mathcal{A}_1$ with no section whose Jacobian is $\mathcal{X}_1 \to \mathcal{A}_1$. We do not know a geometric construction of this family.

Definition 6.9. Let (X, D) be a pair of a projective variety and a $\mathbb{Q}$-divisor D. We say that (X, D) is *KSBA-stable* if:

(1) the pair (X, D) has slc singularities (see e.g. [40]), and
(2) $K_X + D$ is $\mathbb{Q}$-Cartier and ample.

Proposition 6.10. *Let ϵ be a sufficiently small positive rational number. Then every fiber of $(\overline{\mathcal{X}_g^\star}^{\mathrm{vor}}, \epsilon\overline{\Theta}_g^{\mathrm{vor}}) \to \overline{\mathcal{A}}_g^{\mathrm{vor}}$ is a KSBA-stable pair.*

Sketch. In any Mumford construction of second Voronoi type, see Construction 6.5, all fibers have slc singularities, and the canonical bundle $K_X \simeq \mathcal{O}_X$ is trivial. This follows from a mild generalization to affine toric bases of Proposition 3.14, by checking that slices $\mathcal{S}_{(r_1,\ldots,r_k)}$ of the normal fan, for $r_1B_1 + \cdots + r_kB_k$ integral, are integral tilings of $\mathbf{N}_\mathbb{R}$. Indeed, $\mathcal{S}_{(r_1,\ldots,r_k)}$ is, up to translation, the image of the Voronoi decomposition Vor_B under the map $N_\mathbb{R} \colon \mathbf{M}_\mathbb{R} \to \mathbf{N}_\mathbb{R}$ corresponding to $B = \sum_{i=1}^k r_iB_i$. A linear algebra computation verifies the integrality.

The log canonical centers of X are exactly the toric strata of the Mumford construction. Given an effective divisor $D \subset X$, there is an $\epsilon \ll 1$ for which $(X, \epsilon D)$ defines a KSBA-stable pair if and only if D contains no log canonical centers, i.e. toric strata. In fact, in our setting, any $\epsilon \leq 1$ suffices, see [3, Thm. 3.10], generalizing [38, Thm. 17.13].

We claim this property follows from the definition of $\Theta_{(0/1,\ldots,0/1)}$. The key observation is that, for every vertex of a polyhedral face $F \subset \mathrm{Del}_B$, $B \in \mathbb{B}^\circ$, the restriction $\Theta_{(0/1,\ldots,0/1)}|_{Y_F}$ of the theta divisor to the stratum $Y_F \subset X$ is a section of a (toric) line bundle, for which the coefficient of any monomial corresponding a vertex of F is nonzero, i.e. lies in $\mathbb{C}^*$. This property ensures that the restriction of the theta divisor contains no toric strata of Y_F. $\square$

Theorem 6.11 ([4, Thm. 1.2.17])**.** *For $\epsilon \ll 1$, $(\overline{\mathcal{X}_g^\star}^{\mathrm{vor}}, \epsilon\overline{\Theta}_g^{\mathrm{vor}}) \to \overline{\mathcal{A}}_g^{\mathrm{vor}}$ is the universal family over the normalization of the KSBA compactification of the space of KSBA-stable pairs $(X, \epsilon\Theta)$, with X a torsor under a g-dimensional PPAV and $\Theta \subset X$ the theta divisor.*

Sketch. By Proposition 6.10, there is a classifying morphism $c\colon \overline{\mathcal{A}}_g^{\mathrm{vor}} \to \overline{\mathcal{A}}_g^{\Theta}$ where the latter is, by definition, the closure, taken with reduced scheme structure, of the space of pairs $(X, \epsilon\Theta)$ as in Construction 6.6, in the separated DM stack of KSBA-stable pairs [39, 3, 40]. By Zariski's main theorem and the normality of toroidal compactifications, it suffices to check that c is finite.

It is easy to see that c defines a morphism over the Baily–Borel compactification $\overline{A}_g^{\mathrm{BB}}$, e.g. by considering the Albanese variety of the normalization of any component of $(X, \epsilon\Theta)$. So if c contracted some curve, this curve would lie in a fiber of the morphism $\overline{A}_g^{\mathrm{vor}} \to \overline{A}_g^{\mathrm{BB}}$. Any such curve admits an algebraic deformation to a union of 1-dimensional torus orbits—first move the image point in $\overline{A}_g^{\mathrm{BB}}$ to the deepest cusp, then apply the torus action. Thus, c would contract some 1-dimensional toric boundary stratum $(\mathbb{P}^1, 0, \infty) \to \overline{A}_g^{\mathrm{vor}}$. But, for any cone $\mathbb{B} \in \mathfrak{F}_{\mathrm{vor}}$, the combinatorial types of the KSBA-stable fibers over 0 and over $u \in \Delta^* \subset \mathbb{P}^1$ are distinct, by Construction 6.5. It follows that c contracts no algebraic curves, and hence is finite. $\square$

A similar strategy was employed in [6, Thm. 1, Thm. 5.14] to prove the semitoroidality of certain KSBA compactifications of the moduli of polarized K3 surfaces.

6.2. **Algebraicity and projectivity.** We now analyze under what circumstances an extension of the universal family $\mathcal{X}_g \to \mathcal{A}_g$ or $\mathcal{X}_g^\star \to \mathcal{A}_g$ of principally polarized abelian varieties or torsors, by Mumford constructions, are either algebraic or projective.

Proposition 6.12. *Let $\mathbb{B} \subset \mathcal{P}_g^+$ be a rational polyhedral cone and $\mathcal{S}$ be a fan satisfying the hypotheses of Construction 3.11. The corresponding Mumford construction $f\colon X^{\mathrm{univ}}(\mathcal{S}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ is a proper, flat morphism of algebraic spaces.*

Suppose, furthermore, that $f\colon X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ is a polytopal Mumford construction, as in Construction 3.26. Then f is étale-locally projective.

Proof. By replacing $\widetilde{\mathcal{A}}_g$ with a suitable further cover, we may assume that the distinction between $\mathcal{X}_g$ and $\mathcal{X}_g^\star$ is erased. Let $\mathfrak{F}$ be a common refinement of the fans $\Gamma \cdot \mathbb{B}$ and $\mathfrak{F}_{\mathrm{vor}}$ whose support is $\Gamma \cdot \mathbb{B}$, for $\Gamma \subset \mathrm{GL}_g(\mathbb{Z})$ the Levi quotient. Then we have morphisms

$$\widetilde{\mathcal{A}}_g^{\mathbb{B}} \leftarrow \widetilde{\mathcal{A}}_g^{\mathfrak{F}} \to \widetilde{\mathcal{A}}_g^{\mathrm{vor}}$$

and we may pullback the (a priori) analytic family $X^{\mathrm{univ}}(\mathcal{S}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ and the algebraic universal family $\widetilde{\mathcal{X}}_g^{\mathrm{vor}} \to \widetilde{\mathcal{A}}_g^{\mathrm{vor}}$ to produce two families $X^{\mathrm{univ}\prime}(\mathcal{S})$ and $X^{\mathrm{vor}} \to \widetilde{\mathcal{A}}_g^{\mathfrak{F}}$ in the analytic and algebraic categories, respectively.

Taking a common refinement of the fans defining $X^{\mathrm{univ}\prime}(\mathcal{S})$ and X^{vor} we may dominate $X^{\mathrm{univ}\prime}(\mathcal{S})$ and X^{vor} by a common (universal) Mumford construction $\widetilde{X} \to \widetilde{\mathcal{A}}_g^{\widetilde{\mathfrak{F}}}$. Since $\widetilde{X} \to X^{\mathrm{univ}\prime}(\mathcal{S})$ and $\widetilde{X} \to X^{\mathrm{vor}}$ are both toroidal morphisms, we can connect $X^{\mathrm{vor}} \dashrightarrow X^{\mathrm{univ}\prime}(\mathcal{S})$ by a sequence of toric modifications, all of which are algebraic. We deduce that $X^{\mathrm{univ}\prime}(\mathcal{S})$ is an algebraic space. In turn, its contraction $X^{\mathrm{univ}}(\mathcal{S})$ is a proper algebraic space over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$.

Finally, we address the case of a polytopal Mumford construction $f\colon X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$. Then f is analytically-locally projective over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$ because f is a descent of $f_\circ\colon X_\circ^{\mathrm{univ}}(\mathbb{b}) \to T(\mathbb{B})$, which is relatively projective over the maximal open subset $T(\mathbb{B})$ discussed at the end of Construction 3.11, and the map $T(\mathbb{B}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ along which $f_\circ$ descends to f is an étale surjection. The second statement of the proposition is a consequence of the following general fact: If a separated and finitely presented morphism of algebraic spaces $X \to Y$ is analytically-locally projective, then it is étale-locally projective.

The proof follows from Artin approximation. Indeed, the Hilbert scheme $\mathrm{Hilb}_{X/Y}$ is an algebraic space locally of finite presentation over Y by [61, Tag 0D01]. Take a point $p \in Y$. An analytic family of ample divisors $D_U \subset X_U \to U$ over an analytic neighborhood $U \ni p$, may be approximated by an algebraic family of ample divisors, $D'_{U'} \subset X_{U'} \to U'$ over an étale neighborhood $U' \ni p$, which coincides with the restriction of D_U to p. Possibly replacing U' with a smaller, Zariski open neighborhood of $p \in U'$, the divisor $D'_{U'}$ is relatively ample. □

We now consider the much subtler question of when $X^{\mathrm{univ}}(\mathbb{b})$ is *projective* over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$, as opposed to étale-locally projective.

Definition 6.13. Let $\overline{b}, \overline{b}' \in H^0(\mathbb{T}^g, \tfrac{1}{d}\mathbb{Z}\mathrm{PL}/\tfrac{1}{d}\mathbb{Z}\mathrm{L})$. We say that $\overline{b} \sim \overline{b}'$ lie in the same *shift class* if $\overline{b} - \overline{b}'$ lifts to an $\mathbf{M}$-periodic section $b - b'\colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$ of $\tfrac{1}{d}\mathbb{Z}\mathrm{PL}$.

Recall that $d\mathbf{M} \simeq \mathbb{Z}^g = H_1(\mathbb{T}^g, \mathbb{Z})$ in Construction 3.38, while $\mathbf{M} \simeq (\tfrac{1}{d}\mathbb{Z})^g$. A necessary, but in general insufficient, condition for $\overline{b} \sim \overline{b}'$ is that they define the same monodromy bilinear form B via formula (24).

Example 6.14. Let $g = 1$ and consider $\overline{b}, \overline{b}', \overline{b}''$ for which $\mathrm{Bend}(\overline{b}) = 2[\tfrac{0}{1}]$, $\mathrm{Bend}(\overline{b}') = [\tfrac{0}{2}] + [\tfrac{1}{2}]$, and $\mathrm{Bend}(\overline{b}'') = [\tfrac{1}{3}] + [\tfrac{2}{3}]$ as $\mathbb{Z}$-weighted linear combinations of $\tfrac{1}{d}$-integral codimension 1 polytopes, see Definition 3.17. All three define the same monodromy bilinear form $B = 2x^2$. But we have $\overline{b} \sim \overline{b}''$ and $\overline{b} \not\sim \overline{b}'$. See Figure 23. The fundamental issue is that, while we could subtract from $b - b'$ a linear function of slope $\tfrac{1}{2}$ to make it periodic, such a function is not a section of $\tfrac{1}{2}\mathbb{Z}\mathrm{L}$ on $\mathbf{M}_\mathbb{R} \simeq \mathbb{R}$, see Definition 3.25, because it does not have integral slope.

Proposition 6.15. *Let $f\colon X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ and $f'\colon X^{\mathrm{univ}}(\mathbb{b}') \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ be two universal polytopal Mumford Constructions 3.26, for lifts of two cones $\mathbb{b}, \mathbb{b}' \subset H^0(\mathbb{T}^g, \tfrac{1}{d}\mathbb{Z}\mathrm{PL}/\tfrac{1}{d}\mathbb{Z}\mathrm{L})$ into*

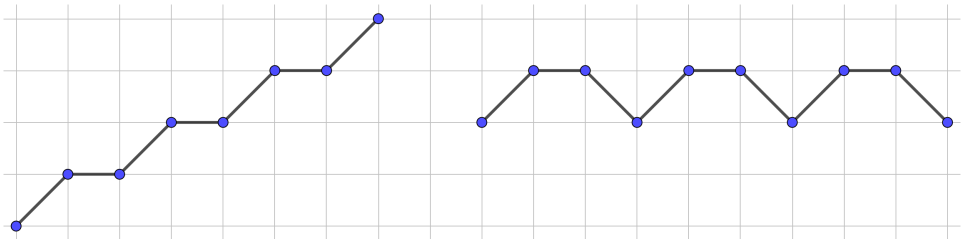


FIGURE 23. Left: lift of section $\overline{b} - \overline{b}'$ with bending $[\frac{0}{2}] - [\frac{1}{2}]$. Right: lift of section $\overline{b} - \overline{b}''$ with bending $2[\frac{0}{3}] - [\frac{1}{3}] - [\frac{2}{3}]$.

$H^0(\mathbf{M}_\mathbb{R}, \frac{1}{d}\mathbb{Z}\mathrm{PL})$, mapping to the same monodromy cone $\mathbb{B} \subset \mathrm{Sym}^2\mathbf{M}^\vee$. Suppose, that for any $B \in \mathbb{B}$, the PL lifts $b, b' \colon \mathbf{M}_\mathbb{R} \to \mathbb{R}$ satisfy the property that $b' - b$ is $\mathbf{M}$-periodic. Then, the canonical analytic polarization on the Mumford construction extends as a relatively ample global section of the relative Picard $\mathrm{Pic}_{X^{\mathrm{univ}}(\mathbb{b})/\widetilde{\mathcal{A}}_g^\mathbb{B}}$ if and only if the same holds for $X^{\mathrm{univ}}(\mathbb{b}')/\widetilde{\mathcal{A}}_g^\mathbb{B}$.

We note that the data of the gluing of the Mumford construction onto the universal family $\widetilde{\mathcal{X}}_g \to \widetilde{\mathcal{A}}_g$ requires the data of lifts of $\mathbb{b}$, $\mathbb{b}'$ into PL functions on $\mathbf{M}_\mathbb{R}$ by Remark 3.39.

Proof. Let $\mathcal{L}$ and $\mathcal{L}'$ be the defining relatively ample line bundles of the two Mumford constructions over the analytic tubular neighborhood $T(\mathbb{B}) \subset Y(\mathbb{B})$ of the deepest toric stratum.

Take a $\mathbf{M}$-periodic, regular refinement $\mathcal{S}$ of the normal fans for $\mathbb{b}$, $\mathbb{b}'$, and let $X^{\mathrm{univ}}_\circ(\mathcal{S}) \to T(\mathbb{B})$ be the corresponding fan Construction 3.11. Let $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}'_\mathcal{S}$ be the pullbacks of $\mathcal{L}$ and $\mathcal{L}'$ to $X^{\mathrm{univ}}(\mathcal{S})$, and define $\mathcal{E} := \mathcal{L}'_\mathcal{S} \otimes \mathcal{L}_\mathcal{S}^{-1}$. Finally, let $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{L}}_\mathcal{S}$, $\widetilde{\mathcal{L}}'_\mathcal{S}$ be the pullbacks of $\mathcal{E}$, $\mathcal{L}_\mathcal{S}$, $\mathcal{L}'_\mathcal{S}$ to the universal cover of $X^{\mathrm{univ}}_\circ(\mathcal{S})$. Then $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{L}}_\mathcal{S}$, $\widetilde{\mathcal{L}}'_\mathcal{S}$ are $\mathbf{M}$-equivariant line bundles. The condition that $b' - b$ is $\mathbf{M}$-periodic implies that we have an $\mathbf{M}$-equivariant isomorphism

$$\widetilde{\mathcal{E}} \simeq \mathcal{O}\Big(\sum_{\substack{\text{rays} \\ \mathbb{R}_{\geq 0} v_i \in \mathcal{S}}} a_{v_i} D_{v_i} \Big),$$

with $a_{v_i} \in \mathbb{Z}$ depending only on the $\mathbf{M}$-equivalence class $\overline{v}_i$ of the ray $\mathbb{R}_{\geq 0} v_i \in \mathcal{S}$. Quotienting, we deduce that the line bundle $\mathcal{E}$ is represented by a finite $\mathbb{Z}$-linear sum $\sum a_{\overline{v}_i} D_{\overline{v}_i}$ of components over the boundary of $X^{\mathrm{univ}}_\circ(\mathcal{S})$. As components over the boundary, $D_{\overline{v}_i}$ descend to algebraic divisors on the algebraic space $X^{\mathrm{univ}}(\mathcal{S}) \to \widetilde{\mathcal{A}}_g^\mathbb{B}$ and thus $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}'_\mathcal{S}$ differ by twisting by a linear combination of vertical divisors, over the boundary of $\widetilde{\mathcal{A}}_g^\mathbb{B}$. So one extends as a section of relative Picard if and only if the other does. Furthermore, the relative ampleness of $\mathcal{L}$ and $\mathcal{L}'$ over the interior $\widetilde{\mathcal{A}}_g$ are equivalent. On the other hand, the relative ampleness of either over the boundary $\widetilde{\mathcal{A}}_g^\mathbb{B} \setminus \widetilde{\mathcal{A}}_g$ is automatic, by construction. $\square$

Proposition 6.16. *Let $f \colon X^{\mathrm{univ}}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^\mathbb{B}$ be a universal polytopal Mumford construction associated to a lift of $\mathbb{b} \subset H^0(\mathbb{T}^g, \frac{1}{d}\mathbb{Z}\mathrm{PL}/\frac{1}{d}\mathbb{Z}\mathrm{L})$, extending (as an algebraic space) the universal family $\widetilde{\mathcal{X}}_g \to \widetilde{\mathcal{A}}_g$ of abelian varieties. Then f is a projective morphism, whenever $b_i(\mathbf{m}) - b_i(-\mathbf{m})$ is $\mathbf{M}$-periodic. Similarly, there is a relatively projective extension $f \colon X^{\mathrm{univ}\star}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g$ of the* *universal family* $\widetilde{\mathcal{X}}_g^\star \to \widetilde{\mathcal{A}}_g$ *of abelian torsors when* $\overline{b} \sim \overline{b}^{\rm vor}$ *lie in the same shift class, for any* $B \in \mathbb{B}$*, and for* $\overline{b}^{\rm vor}$ *defined as in (33).*

Proof. To prove the extension result for $\widetilde{\mathcal{X}}_g$, we follow the proof strategy of Proposition 6.15: We may pass to a smooth (-1)-symmetric common refinement $X_\circ^{\rm univ}(\mathcal{S}) \to T(\mathbb{B})$ of the Mumford constructions for $b_i(\mathbf{m})$ and $b_i(-\mathbf{m})$. The pulled back line bundles $\mathcal{L}_S$, $\mathcal{L}'_{\mathcal{S}}$ associated to $b_i(\mathbf{m})$, $b_i(-\mathbf{m})$ are interchanged by the (-1)-involution: $(-1)^*\mathcal{L}_{\mathcal{S}} \simeq \mathcal{L}'_{\mathcal{S}}$. If $b_i(\mathbf{m}) - b_i(-\mathbf{m})$ is $\mathbf{M}$-periodic, we may conclude that $\mathcal{L}'_{\mathcal{S}} \simeq \mathcal{L}_{\mathcal{S}}(\sum a_{\overline{v}_i} D_{\overline{v}_i})$ differ by twisting by vertical divisors. It follows that $(\mathcal{L}_{\mathcal{S}})^{\otimes 2}$ defines an algebraic extension of $(\mathcal{L}_{\widetilde{\mathcal{X}}_g})^{\otimes d}$ where $\mathcal{L}_{\widetilde{\mathcal{X}}_g} \in \mathrm{Pic}_{\widetilde{\mathcal{X}}_g/\widetilde{\mathcal{A}}_g}(\widetilde{\mathcal{A}}_g)$ is a tensor square of a (-1)-symmetric local lift of the principal polarization.

The case of extending $\widetilde{\mathcal{X}}_g^\star$ is similar, but again Proposition 6.15 does not directly apply, since we are gluing onto the universal abelian torsor. The extension of $\widetilde{\mathcal{X}}_g^\star$ by the Mumford construction of second Voronoi type (Construction 6.6) is relatively projective. Replacing $\mathbb{B}$ with a common refinement of $\mathbb{B}$ and $\mathfrak{F}_{\rm vor}$, we may assume that $\mathbb{B}$ is contained in a second Voronoi cone. Choosing lifts $b^{\rm vor}$ as in (33), defines a local analytic section of $\overline{\mathcal{X}_g^\star}^{\rm vor}$ near the boundary stratum associated to a second Voronoi cone. The hypothesis that $\overline{b} \sim \overline{b}^{\rm vor}$, for all $B \in \mathbb{B}$ and argument of Proposition 6.15 show that, with respect to the chosen local origin section of $\widetilde{\mathcal{X}}_g^\star$, and a well chosen lift of $\mathbb{b}$, there is a gluing $X^{\rm univ\star}(\mathbb{b}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}}$ of $X^{\rm univ}(\mathbb{b}) \to T(\mathbb{B})$ and $\widetilde{\mathcal{X}}_g^\star \to \widetilde{\mathcal{A}}_g$ for which the ample line bundle on the former extends to a section in $\mathrm{Pic}_{X^{\rm univ\star}(\mathbb{b})/\widetilde{\mathcal{A}}_g^{\mathbb{B}}}(\widetilde{\mathcal{A}}_g^{\mathbb{B}})$.

Thus, in either case $\widetilde{\mathcal{X}}_g$ or $\widetilde{\mathcal{X}}_g^\star$, the canonical polarization on the Mumford construction extends to a relatively ample section of relative Picard over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$.

After passing to some further tensor power, we may lift to an element $\mathcal{E}$ of the (algebraic) Picard group of $X^{\rm univ}(\mathbb{b})$ or $X^{\rm univ\star}(\mathbb{b})$, which is relatively very ample. Pushing forward, we get a vector bundle $f_*\mathcal{E}$ over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$ over the étale site, and therefore over the Zariski site. It follows that the projectivization of $(f_*\mathcal{E})^\vee$ is relatively projective over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$. Thus, $X^{\rm univ}(\mathbb{b})$ or $X^{\rm univ\star}(\mathbb{b})$ admit closed, algebraic embeddings into a projective space over $\widetilde{\mathcal{A}}_g^{\mathbb{B}}$. The result follows. ☐

Projectivity criteria for extensions of $\widetilde{\mathcal{X}}_g$ should be compared to [17, Ch. VI]. Following a standard toric construction, the polytope Γ of Section 3.3 defines a convex PL function $\tilde{\phi}\colon \mathcal{S} \to \mathbb{R}$ on the normal fan $\mathcal{S}$ of Section 3.1. The function $\tilde{\phi}$ should be an "admissible homogeneous principal polarization function" as in [17, Ch. VI, Def. 1.5], with our conditions on the shift class of b_i related to Def. 1.5.(vi) of *loc.cit.* Our projectivity results should then follow from [17, Ch. VI, Thm. 1.13], though translating between the language used here and that in *loc.cit.* is somewhat involved.

Corollary 6.17. *Suppose that* $\mathscr{H}$ *is a hyperplane arrangement for the regular matroid* $\underline{R}$*, for which the parallel hyperplanes normal to* $\vec{x}_i$ *are* $H_i^{(j)} := \vec{x}_i(\mathbf{m}) \in \epsilon_{ij} + \mathbb{Z}$*. Then the extension*

$$X^{\rm univ\star}(\underline{R}, \mathscr{H}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$$

of $\widetilde{\mathcal{X}}_g^\star \to \widetilde{\mathcal{A}}_g$ *is projective whenever* $\sum_j \epsilon_{ij} = 0 \in \mathbb{Q}/\mathbb{Z}$ *for all* $i = 1, \dots, k$ *and the extension*

$$X^{\mathrm{univ}}(\underline{R}, \mathscr{H}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$$

of $\widetilde{\mathcal{X}}_g \to \widetilde{\mathcal{A}}_g$ *is projective whenever* $\sum_j (\epsilon_{ij} + \frac{1}{2}) = 0 \in \mathbb{Q}/\mathbb{Z}$ *for all* $i = 1, \dots, k$.

Proof. Let r_i be the number of hyperplanes $H_i^{(j)} \in \mathscr{H}$ normal to $\vec{x}_i$.

For $\mathbb{B}_{\underline{R}} \in \mathfrak{F}_{\mathrm{vor}}$ a matroidal cone, the Delaunay decomposition as in Construction 6.5 is the unshifted hyperplane arrangement (27). Thus, it follows from Proposition 6.16 that $\mathscr{H}$ defines a projective extension of $\widetilde{\mathcal{X}}_g^\star \to \widetilde{\mathcal{A}}_g$ whenever the section $\overline{b}_i \in H^0(\mathbb{T}^g, \frac{1}{d}\mathbb{Z}\mathrm{PL}, \frac{1}{d}\mathbb{Z}\mathrm{L})$ bending along $H_i^{(j)} := \{\vec{x}_i(\mathbf{m}) \in \epsilon_{ij} + \mathbb{Z}\}$ lies in the same shift class as $\overline{b}_i^o \in H^0(\mathbb{T}^g, \frac{1}{d}\mathbb{Z}\mathrm{PL}, \frac{1}{d}\mathbb{Z}\mathrm{L})$ which bends r_i times along the unshifted hyperplane $\vec{x}_i(\mathbf{m}) \in 0 + \mathbb{Z}$. Equivalently,

$$\int_0^1 \int_{0^-}^x (-r_i\delta_0 + \textstyle\sum_{j=1}^{r_i} \delta_{\epsilon_{ij}})\, dy\, dx \in \mathbb{Z} \tag{34}$$

where δ_p denotes the Dirac delta function at p. Now (35) holds if and only if $\sum_{j=1}^{r_i} \epsilon_{ij} \in \mathbb{Z}$.

For the case of extending $\widetilde{\mathcal{X}}_g \to \widetilde{\mathcal{A}}_g$ projectively, we observe that an arrangement which bends r_i times along the half-shifted hyperplane $\vec{x}_i(\mathbf{m}) \in \frac{1}{2} + \mathbb{Z}$ satisfies the hypotheses of Proposition 6.16 by lifting to a $\frac{1}{2}\mathbb{Z}\mathrm{PL}$ function b_i' which is identically zero in a neighborhood of the origin of $\mathbf{M}_{\mathbb{R}}$. So b_i' defines a relatively projective extension of the universal family $\widetilde{\mathcal{X}}_g \to \widetilde{\mathcal{A}}_g$. Then by Proposition 6.15, it suffices to understand when $\overline{b}_i' \sim \overline{b}_i$ lie in the same shift class, i.e.

$$\int_0^1 \int_{0^-}^x (-r_i\delta_{\frac{1}{2}} + \textstyle\sum_{j=1}^{r_i} \delta_{\epsilon_{ij}})\, dy\, dx \in \mathbb{Z}. \tag{35}$$

This holds exactly when $\sum_{j=1}^{r_i} (\epsilon_{ij} + \frac{1}{2}) \in \mathbb{Z}$. □

7. Proof of Theorem 1.1

Our goal is to prove Theorem 1.1, and leverage Proposition 6.12 to prove more algebraic formulations of the results therein, see Theorem 7.1 and Corollary 7.2 below.

Proof of Theorem 1.1. Let $f^*\colon X^* \to (\Delta^*)^k$ be a family of PPAVs which is matroidal with respect to the snc extension $(\Delta^*)^k \hookrightarrow \Delta^k$ (Def. 4.9). Then there are integers $r_i > 0$ for which the monodromy bilinear forms about $\{u_i = 0\}$ are $r_i B_i$ where $B_i = \mathbf{x}_i^2$ is an integral generator of the matroidal cone $\mathbb{B}_{\underline{R}}$ of the corresponding regular matroid $\underline{R}$. Constructions 4.16 and 3.26, see also Notation 4.19, give a universal Mumford degeneration

$$f^{\mathrm{univ}}\colon X^{\mathrm{univ}}(\underline{R}, \mathscr{H}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$$

on a transversely shifted hyperplane arrangement for the associated regular matroid $\underline{R}$, which has exactly r_i bending loci in $\mathbb{T}^g$, with bending parameter 1, along hyperplanes normal to $\mathbf{x}_i$.

The monodromies about the boundary divisors of $\widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$ are exactly $r_i B_i$ and by Proposition 2.27, the classifying morphism $(\Delta^*)^k \to \mathcal{A}_g$ will (lift and) extend to $\Delta^k \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$. Pulling back $X^{\mathrm{univ}}(\underline{R}, \mathscr{H})$ along the extension of the classifying morphism to Δ^k produces the desired extension $f\colon X \to \Delta^k$. It has smooth total space and nodal singularities, because f^{univ} is locally trivial along the deepest toroidal stratum of $\widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$ and so smoothness, resp. nodality, of X^{univ}, resp. f^{univ} (see Proposition 5.3), implies smoothness, resp. nodality, of the restrictions X, resp. f, to the transversal slice Δ^k to this deepest toroidal stratum.

The condition that $f\colon X \to \Delta^k$ be strictly nodal follows from the condition $r_i \geq 2$ by Proposition 5.5.

Finally, we address the K-triviality of X. It suffices, by Proposition 3.14, to show that the slice $\mathcal{S}_{(1,\ldots,1)}$ of the normal fan has integral vertices. Indeed, this holds for any shifted matroidal degeneration, as each top-dimensional cell in $\mathcal{S}_{(1,\ldots,1)}$ is a Minkowski sum of segments $\sum_{i\in I_v}[0,\mathbf{x}_i] \subset \mathbf{N}_{\mathbb{R}}$ corresponding to hyperplanes $H_i \in \mathscr{H}$ meeting at the dual vertex $v \in \bigcap_{i\in I_v} \overline{H}_i$ of the arrangement in $\mathbf{M}_{\mathbb{R}}/\mathbf{M}$. (For a transversely shifted arrangement, these Minkowski sums are integral-affine unit cubes). We deduce the first part of Theorem 1.1.

The second part of Theorem 1.1 follows from the existence of algebraic, transversely shifted matroidal degenerations with specified monodromies, see Corollary 7.2 below. □

Theorem 7.1. *Let $f^*\colon X^* \to Y^*$ be a projective family of PPAVs over a base $Y^* = Y \setminus D$ for $D \subset Y$ an snc divisor in a smooth quasiprojective variety Y. Let $0 \in D$ and assume the local monodromy bilinear forms B_i about the components $D_i \ni 0$ are $r_i\mathbf{x}_i^2$ for an integral realization $i \mapsto \mathbf{x}_i \in \mathbf{M}^\vee$ of a regular matroid $\underline{R}$, where $\mathbf{M} \simeq \mathrm{gr}_0^W H_1(X_t,\mathbb{Z})$ for t near 0. Up to passing to an étale neighborhood of $0 \in Y$, there is a flat, projective, D-nodal, relatively K-trivial extension $f\colon X \to Y$, which is furthermore strictly D-nodal when all $r_i \geq 2$.*

Proof. We have a classifying map $Y^* \to \mathcal{A}_g$ and by the hypothesis on monodromy, we have, étale-locally about $0 \in Y$, an extension and lift $Y \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$, $\mathbb{B}_{\underline{R}} := \mathbb{R}_{\geq 0}\{B_1,\ldots,B_k\}$, e.g. because this lift exists analytically-locally about $0 \in Y$ (cf. Prop. 2.27). The morphism $Y \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$ is algebraic, for instance by Borel algebraicity. Pulling back the family $X^{\mathrm{univ}}(\underline{R},\mathscr{H}) \to \widetilde{\mathcal{A}}_g^{\mathbb{B}_{\underline{R}}}$, which is algebraic and étale-locally projective by Proposition 6.12, we deduce the result (the nodality and relative K-triviality of the extension follow as in the proof above). □

Corollary 7.2. *Let $\underline{R}$ be a regular matroid of rank g on a k element set and let $(r_1,\ldots,r_k) \in \mathbb{N}^k$. There exists a projective morphism $f\colon X \to Y$ of smooth quasiprojective varieties, $k = \dim Y$, $g + k = \dim X$, an snc divisor $D \subset Y$, and a zero-dimensional stratum $0 \in D$, satisfying the following conditions:*

(1) *The monodromies about the components $D_i \ni 0$ of D are of the form $B_i = r_i\mathbf{x}_i^2$ and generate the matroidal cone $\mathbb{B}_{\underline{R}}$ (Defs. 2.6, 4.6).*
(2) *The morphism $f\colon X \to Y$ is a transversely shifted matroidal degeneration on the matroid $\underline{R}$ near $0 \in Y$ (Def. 4.18), and the restriction of f to $Y^* := Y \setminus D$ is a family of principally polarized abelian varieties of dimension g.*
(3) *The morphism f is, up to shrinking Y, a D-nodal morphism, which is furthermore strictly D-nodal if $r_i \geq 2$ for all $i = 1,\ldots,k$ (Def. 5.1).*

Proof. By Proposition 4.10, there is a smooth quasiprojective variety Y, snc divisor $D \subset Y$, zero-stratum $0 \in D$, and family $f^*\colon X^* \to Y^*$ of PPAVs over $Y^* = Y \setminus D$, whose monodromies about the components $D_i \ni 0$ are given by $r_i \mathbf{x}_i^2$. Applying Theorem 7.1 and passing to an étale chart about 0, we produce a projective extension $f\colon X \to Y$ with the desired properties. $\square$

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