# A note on descent for algebraic stacks

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## 1. Introduction

Let $S' \to S$ be a morphism of affine schemes, faithfully flat and locally of finite presentation. By a theorem of Grothendieck, the functor $X \mapsto X \times_S S'$ defines an equivalence of categories between the category of $S$-schemes $X$ and the category of pairs $(X', \phi)$ where $X'$ is an $S'$-scheme and $\phi$ a descent datum for $X'$ over $S'$ such that $X'$ admits an open covering by affine schemes which are stable under $\phi$. In case $S = \mathrm{Spec}(k)$, $S' = \mathrm{Spec}(k')$ and the morphism $S' \to S$ corresponds to a finite Galois extension of fields $k \subset k'$, this is known as Galois descent, and due to Weil.

The goal of this note is to prove a similar statement for algebraic stacks. In the case of stacks, the analogue of the aforementioned descent-theory is a notion called 2-*descent*, which seems to be due to Duskin [Dus89]. It turns out that, with respect to a morphism of schemes $S' \to S$ which is smooth and surjective, *every* 2-*descent datum for an algebraic stack is effective.* More precisely, we have the following result. For a scheme $S$, let $(\mathrm{Sch}/S)_{fppf}$ be the big fppf site of $S$ as in [Stacks, Tag 021S]; a *stack over* $S$ is a stack in groupoids $\mathcal{X} \to (\mathrm{Sch}/S)_{fppf}$ over $(\mathrm{Sch}/S)_{fppf}$, see [Stacks, Tag 0304].

Theorem 1.1. *Let $S' \to S$ be a faithfully flat morphism of schemes locally of finite presentation, and let $\mathcal{X}'$ be a stack over $S'$. Let $(\phi, \psi)$ be a 2-descent datum for the stack $\mathcal{X}'$ over $S'$, see Definition 3.1. Then $(\phi, \psi)$ is effective. That is, there exists a stack $\mathcal{X}$ over $S$, an isomorphism of stacks over $S'$*

$$\rho \colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}',$$

*and a 2-isomorphism $\chi \colon p_2^* f \circ \mathrm{can} \Rightarrow \phi \circ p_1^* f$ as in the following diagram:*

$$\begin{array}{ccc} p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\mathrm{can}} & p_2^*(\mathcal{X} \times_S S') \\ \big\downarrow{\scriptstyle p_1^*\rho} & \swArrow & \big\downarrow{\scriptstyle p_2^*\rho} \\ p_1^*\mathcal{X}' & \xrightarrow{\phi} & p_2^*\mathcal{X}', \end{array} \tag{1.1}$$

*such that the natural compatibility between $\chi$ and $\psi$ is satisfied. Moreover, if $S' \to S$ is smooth, then $\mathcal{X}'$ is an algebraic stack over $S'$ if and only if $\mathcal{X}$ is an algebraic stack over $S$. Finally, if $S' \to S$ is étale, then $\mathcal{X}'$ is a Deligne–Mumford stack over $S'$ if and only if $\mathcal{X}$ is a Deligne–Mumford stack over $S$.*

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Note that even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result (cf. Corollary 3.4). Of course, in some sense these results are not surprising: the descended stack $\mathcal{X}$ is obtained by defining $\mathcal{X}(T)$ as the groupoid of objects of $\mathcal{X}'(T \times_S S')$ equipped with a descent datum relative to the 2-descent datum of $\mathcal{X}'$, for any scheme $T$ over $S$. More precisely, the first assertion in the above theorem follows from the fact that the 2-fibred category $\underline{Stack}_S$ over $(\mathrm{Sch}/S)_{fppf}$, whose fibre over $U \in (\mathrm{Sch}/S)_{fppf}$ is the category $\underline{Stack}(U)$ of stacks over $U$, is a 2-stack over $S$ (see e.g. [Bre94, Example 1.11.(1)]). The other two assertions follow from the fact that the property of a stack of being algebraic (resp. Deligne–Mumford) is local for the smooth (resp. étale) topology, see Lemma 3.3. For details, see Section 3.

In case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group $\Gamma$, then for a stack $\mathcal{X}'$ over $S'$, one can reformulate the notion of 2-descent datum for $\mathcal{X}'$ over $S'$ in terms of an action of $\Gamma$ on $\mathcal{X}'$ over the action of $\Gamma$ on $S'$ over $S$, as in the classical case. To explain this, for an element $\sigma \in \Gamma$, define ${}^\sigma\mathcal{X}'$ as the pull-back of $\mathcal{X}'$ along $\sigma\colon S' \to S'$.

Definition 1.2. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group $\Gamma$. Let $\mathcal{X}'$ be a stack over $S'$. A* Galois 2-descent datum *consists of:*

*(1) a family of 1-isomorphisms $f_\sigma\colon {}^\sigma\mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$ $(\sigma \in \Gamma)$;*
*(2) a family of 2-isomorphisms $\psi_{\sigma,\tau}\colon f_\sigma \circ {}^\sigma(f_\tau) \implies f_{\sigma\tau}$ $(\sigma, \tau \in \Gamma)$;*

*such that for each $\sigma, \tau, \gamma \in \Gamma$, the diagram of 2-morphisms*

$$\begin{array}{ccc} f_\sigma \circ {}^\sigma f_\tau \circ {}^{\sigma\tau} f_\gamma & \xRightarrow{({}^{\sigma\tau}f_\gamma)^*(\psi_{\sigma,\tau})} & f_{\sigma\tau} \circ {}^{\sigma\tau} f_\gamma \\ \Downarrow {\scriptstyle f_{\sigma*}({}^\sigma\psi_{\tau,\gamma})} & & \Downarrow {\scriptstyle \psi_{\sigma\tau,\gamma}} \\ f_\sigma \circ {}^\sigma f_{\tau\gamma} & \xRightarrow{\psi_{\sigma,\tau\gamma}} & f_{\sigma\tau\gamma} \end{array}$$

*is commutative.*

One can show that to give a Galois 2-descent datum on $\mathcal{X}'$ over $S'$ is to give a group action (in the sense of [Rom05]) of $\Gamma$ on $\mathcal{X}'$ as a stack over $S$, such that for each $\sigma \in \Gamma$, the composition $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ agrees with the composition $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$; this is also equivalent to giving 2-descent datum for $\mathcal{X}'$ over $S'$, see Lemma 3.5. As a corollary of Theorem 1.1, one therefore obtains:

Theorem 1.3. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group $\Gamma$. Let $\mathcal{X}'$ be an algebraic stack over $S'$, equipped with a Galois 2-descent datum $(f_\sigma\ (\sigma \in \Gamma),\ \psi_{\sigma,\tau}\ (\sigma, \tau \in \Gamma))$. There exists an algebraic stack $\mathcal{X}$ over $S$ and an isomorphism $\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ of stacks over $S'$. The stack $\mathcal{X}$ is Deligne–Mumford if and only if $\mathcal{X}'$ is.*

Observe that the statement in Theorem 1.3 can be made a bit more precise. Namely, with notation and assumptions as in the theorem, there exists an isomorphism of stacks $\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ over $S'$ as well as a family of 2-isomorphisms $\chi_\sigma\colon \rho \circ \mathrm{can} \Longrightarrow f_\sigma \circ {}^\sigma\rho$ for $\sigma \in \Gamma$ as in the following diagram:

$$\begin{array}{ccc} {}^\sigma(\mathcal{X} \times_S S') & \xrightarrow{\mathrm{can}} & \mathcal{X} \times_S S' \\ \downarrow & \swarrow\!\!\!\!\Downarrow & \downarrow \\ {}^\sigma\mathcal{X}' & \longrightarrow & \mathcal{X}', \end{array}$$

such that the obvious compatibility conditions are satisfied.

Example 1.4. *Let $k$ be a field and let $k \subset k'$ be a degree two field extension; one may think of $\mathbb{R} \subset \mathbb{C}$ or $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ for a prime power $q$. Let $\sigma \in \mathrm{Gal}(k'/k)$ be the generator of $\mathrm{Gal}(k'/k)$. Let $\mathcal{X}'$ be a stack over $k'$ equipped with a 1-isomorphism $\sigma\colon \mathcal{X}' \to \mathcal{X}'$ and a 2-isomorphism $F\colon \sigma^2 \Longrightarrow \mathrm{id}_{\mathcal{X}'}$ between $\sigma^2$ and the identity functor, such that $\sigma$ commutes with the functor $(\mathsf{Sch}/k') \to (\mathsf{Sch}/k')$ defined as $T \mapsto {}^\sigma T = T \times_{k',\sigma} k'$, and such that for each $x \in \mathcal{X}'(T)$, $T \in (\mathsf{Sch}/k')$, the isomorphism $F(x)\colon \sigma^2(x) \to x$ lies over the canonical isomorphism of schemes ${}^\sigma({}^\sigma T) \to T$. One obtains the descended stack $\mathcal{X}$ over $k$ by defining, for $T \in (\mathsf{Sch}/k)$, $\mathcal{X}(T)$ as the groupoid of pairs $(x,\varphi)$ with $x \in \mathcal{X}'(T_{k'})$ and $\varphi\colon x \to \sigma(x)$ an isomorphism such that the composition*

$$x \xrightarrow{\varphi} \sigma(x) \xrightarrow{{}^\sigma\varphi} \sigma^2(x) \xrightarrow{F} x$$

*is the identity. There is a natural isomorphism $\mathcal{X} \times_k k' \cong \mathcal{X}'$ of stacks over $k'$.*

## 2. Descending schemes

Let

$$p\colon S' \to S$$

be a morphism of schemes which is faithfully flat and locally of finite presentation. We get a diagram

$$S'' := S' \times S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \to S,$$

and if $S''' = S' \times_S S' \times_S S'$, we can extend this to the diagram

$$S''' \;\substack{\to\\ \to\\ \to}\; S'' \rightrightarrows S' \to S$$

where the three arrows $S''' \to S''$ are $p_{12}$, $p_{13}$ and $p_{23}$.

Let $X'$ be a scheme over $S'$. Define

$$p_i^* X' = X' \times_{S',p_i} S'', \quad p_{jk}^* p_i^* X' = \left(p_i^* X'\right) \times_{S'',p_{jk}} S'''$$

and note that

$$p_{jk}^* p_i^* X' = \left(p_i^* X'\right) \times_{S'',p_{jk}} S''' = \left(p_i \circ p_{jk}\right)^* X'.$$

Recall that a *descent datum* for $X'/S'$ is an $S''$-isomorphism

$$\phi\colon p_1^* X' \xrightarrow{\sim} p_2^* X'$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} p_{12}^*p_1^*X' & \xrightarrow{p_{12}^*\phi} & p_{12}^*p_2^*X' & = & p_{23}^*p_1^*X' & \xrightarrow{p_{23}^*\phi} & p_{23}^*p_2^*X' \\ \| & & & & & & \| \\ p_{13}^*p_1^*X' & & & \xrightarrow{p_{13}^*\phi} & & & p_{13}^*p_2^*X'. \end{array}$$

In other words, one requires that

$$p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi \quad \text{as morphisms} \quad p_{12}^*p_1^*X' \to p_{13}^*p_2^*X'.$$

Theorem 2.1 (Grothendieck). *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of affine schemes. The functor $X \mapsto p^*X$ defines an equivalence of categories between the category of $S$-schemes $X$ and the category of pairs $(X', \phi)$ where $X'$ is an $S'$-scheme and $\phi$ a descent datum for $X'/S'$ such that $X'$ admits an open covering by affine schemes stable under $\phi$.*

Next, recall how to make this explicit in case $S' \to S$ is a finite faithfully flat morphism of schemes which is a Galois covering with Galois group $\Gamma$. For instance, $S$ could be the spectrum of a field $k$, $S'$ the spectrum of a finite field extension $k' \supset k$, and $\Gamma$ the Galois group of $k'/k$. Let $X'$ be a scheme over $S'$ and call a *Galois descent datum* any set of isomorphisms

$$f_\sigma\colon {}^\sigma X' \xrightarrow{\sim} X'$$

of schemes over $S'$, for $\sigma \in \Gamma$, satisfying the condition that

$$f_{\sigma\tau} = f_\sigma \circ {}^\sigma(f_\tau) \quad \text{as isomorphisms} \quad {}^{\sigma\tau}X' \xrightarrow{\sim} {}^\sigma X' \xrightarrow{\sim} X', \quad \forall \sigma, \tau \in \Gamma.$$

An action of $\Gamma$ on $X'$ as a scheme over $S$ is said to be *compatible with the action of* $\Gamma$ *on* $S'$ *over* $S$ if for each $\sigma \in \Gamma$, the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\sigma} & S'. \end{array}$$

Lemma 2.2. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group $\Gamma$, and let $X'$ be a scheme over $S'$. To give a descent datum for $X'$ over $S'$ is to give a Galois descent datum for $X'$ over $S'$. These notions are further equivalent to giving an action of $\Gamma$ on $X'$ compatible with the action of $\Gamma$ on $S'$ over $S$.*

Proof. This is well-known; see e.g. [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4]. □

## 3. Descending algebraic stacks

Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids on $S'$, in the sense of [Stacks, Tag 0304]. Let

$$S'''' = S' \times_S S' \times_S S' \times_S S';$$

it is equipped with four projections

$$r_i\colon S'''' \to S'. \tag{3.1}$$

Similarly, $S'''$ is equipped with three projections $q_i\colon S''' \to S'$. Note that there are canonical isomorphisms

$$p_{12}^* p_1^* \mathcal{X}' = (p_1 \circ p_{12})^* \mathcal{X}' = q_1^* \mathcal{X}'.$$

Similarly, there are canonical isomorphisms

$$p_{123}^* p_{12}^* p_1^* = (p_1 \circ p_{12} \circ p_{123})^* = r_1^* \mathcal{X}',$$

of algebraic stacks on $S'$. One has similar isomorphisms relating the other $p_{ijk}^* p_{\alpha\beta}^* p_\nu^* \mathcal{X}'$ with $r_\mu^* \mathcal{X}'$, for $i, j, k \in \{1, 2, 3, 4\}$, $\alpha, \beta \in \{1, 2, 3\}$, $\nu \in \{1, 2\}$ and $\mu \in \{1, 2, 3, 4\}$.

Consider an isomorphism of $S''$-stacks (i.e. an equivalence of $\mathrm{Sch}/S''$-categories):

$$\phi\colon p_1^* \mathcal{X}' \to p_2^* \mathcal{X}',$$

and let $\psi$ be a 2-morphism

$$\psi\colon p_{23}^* \phi \circ p_{12}^* \phi \Rightarrow p_{13}^* \phi,$$

which we may picture as the 2-morphism $\Rightarrow$ in the following diagram:

$$\begin{array}{ccccccc} p_{12}^* p_1^* \mathcal{X}' & \xrightarrow{p_{12}^*\phi} & p_{12}^* p_2^* \mathcal{X}' & = & p_{23}^* p_1^* \mathcal{X}' & \xrightarrow{p_{23}^*\phi} & p_{23}^* p_2^* \mathcal{X}' \\ \| & & & \swarrow \psi & & & \| \\ p_{13}^* p_1^* \mathcal{X}' & & & \xrightarrow{\quad p_{13}^*\phi \quad} & & & p_{13}^* p_2^* \mathcal{X}'. \end{array} \tag{3.2}$$

Consider the four maps

$$p_{123}, p_{124}, p_{134}, p_{234}\colon S'''' \to S''',$$

and note that

$$p_{123}^* \left(p_{23}^* \phi \circ p_{12}^* \phi\right) = p_{123}^* p_{23}^* \phi \circ p_{123}^* p_{12}^* \phi = \pi_{23}^* \phi \circ \pi_{12}^* \phi, \quad \text{and}$$
$$p_{123}^* p_{13}^* \phi = \pi_{13}^* \phi,$$

where

$$\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34}\colon S'''' \to S''$$

are the canonical morphisms. For $i, j, k \in \{1, 2, 3, 4\}$ with $i < j < k$, define

$$\psi_{ijk} := p_{ijk}^* \psi.$$

For instance, pulling back $\psi$ along $p_{123}$ gives a 2-morphism

$$\psi_{123} = p_{123}^* \psi\colon \pi_{23}^* \circ \pi_{12}^* \phi \Rightarrow \pi_{13}^* \phi.$$

Similarly, we obtain 2-morphisms

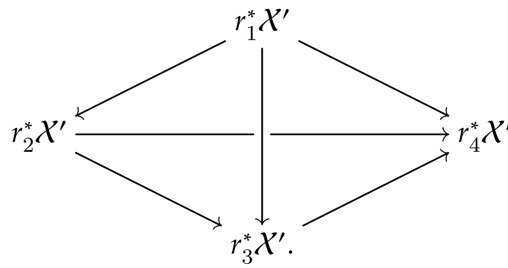
$$\psi_{124}\colon \pi_{24}^* \phi \circ \pi_{12}^* \phi \Rightarrow \pi_{14}^* \phi,$$
$$\psi_{134}\colon \pi_{34}^* \phi \circ \pi_{13}^* \phi \Rightarrow \pi_{14}^* \phi,$$
$$\psi_{234}\colon \pi_{34}^* \phi \circ \pi_{23}^* \phi \Rightarrow \pi_{24}^* \phi.$$

Moreover, observe that under $p_{123}$, diagram (3.2) pulls back to the diagram

$$
\begin{array}{ccccccc}
r_1^*\mathcal{X}' & \xrightarrow{\pi_{12}^*\phi} & r_2^*\mathcal{X}' & = & r_2^*\mathcal{X}' & \xrightarrow{\pi_{23}^*\phi} & r_3^*\mathcal{X}' \\
\| & & & \Swarrow & & & \| \\
r_1^*\mathcal{X}' & & & \xrightarrow{\pi_{13}^*\phi} & & & r_3^*\mathcal{X}',
\end{array}
\tag{3.3}
$$

in which the 2-morphism $\Rightarrow$ is the 2-morphism $\psi_{123}$ defined above (and with $r_i$ is as in (3.1)). Using pull-backs by the other three $p_{ijk}\colon S'''' \to S'''$, we thus obtain four triangles, that we may put together to form the following tetrahedron:

(3.4)



Definition 3.1. *Let $p\colon S' \to S$ be a faithfully flat locally finitely presented morphism of schemes. Let $\mathcal{X}'$ be a stack in groupoids over $S'$. A* 2-descent datum *for $\mathcal{X}'$ over $S'$ consists of:*

*(1) an isomorphism of stacks (i.e. an equivalence of categories)*

$$\phi\colon p_1^*\mathcal{X}' \to p_2^*\mathcal{X}'$$

*over $S''$;*

*(2) a 2-isomorphism*

$$\psi\colon p_{23}^*\phi \circ p_{12}^*\phi \Rightarrow p_{13}^*\phi$$

*as in diagram (3.2);*

*such that the following condition is satisfied: the 2-morphisms $\psi_{ijk}$ between the several compositions in diagram (3.4) are compatible, in the sense that the following diagram of 2-morphisms commutes:*

$$
\begin{array}{ccc}
\pi_{34}^*\phi \circ \pi_{23}^*\phi \circ \pi_{12}^*\phi & \xRightarrow{(\pi_{34}^*\phi)_*(\psi_{123})} & p_{34}^*\phi \circ p_{13}^*\phi \\
\Big\Downarrow {\scriptstyle (\pi_{12}^*\phi)^*(\psi_{234})} & & \Big\Downarrow {\scriptstyle \psi_{134}} \\
p_{24}^*\phi \circ p_{12}^*\phi & \xRightarrow{\psi_{124}} & p_{14}^*\phi.
\end{array}
$$

This gives the following result.

Proposition 3.2 (Breen). *Let $(\phi, \psi)$ be a 2-descent datum for the stack $\mathcal{X}'$ over $S'$. Then there exists a stack $\mathcal{X}$ over $S$, an isomorphism*

$$\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$$

*of stacks over $S'$, and a 2-isomorphism $\chi\colon p_2^*\rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^*\rho$ as in diagram*

$$
\begin{array}{ccc}
p_1^*(\mathcal{X} \times_S S') & \xrightarrow{\mathrm{can}} & p_2^*(\mathcal{X} \times_S S') \\
\Big\downarrow p_1^*\rho & \swarrow\!\!\!\!\swarrow & \Big\downarrow p_2^*\rho \\
p_1^*\mathcal{X}' & \xrightarrow{\phi} & p_2^*\mathcal{X}',
\end{array} \tag{3.5}
$$

*such that the natural compatibility condition between $\chi$ and $\psi$ is satisfied.*

Proof. This follows from [Bre94, Example 1.11.(i)]. □

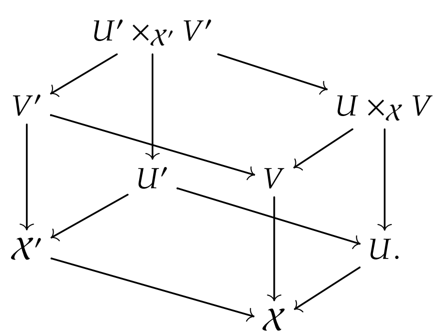
To prove Theorem 1.1, we recall that any stack which is smooth locally algebraic, is algebraic. More precisely, we recall the following well-lemma, which should be well-known but which we include for convenience of the reader.

Lemma 3.3. *Let $S$ be a scheme. The following assertions are true.*

*(1) Let $\pi\colon \mathcal{X}' \to \mathcal{X}$ be a representable, smooth and surjective morphism of stacks in groupoids over $S$. If $\mathcal{X}'$ is algebraic, then $\mathcal{X}$ is algebraic. If in addition $\pi$ is étale and $\mathcal{X}'$ is Deligne–Mumford, then $\mathcal{X}$ is Deligne–Mumford.*

*(2) Let $S' \to S$ be a smooth surjective morphism of schemes, let $\mathcal{X}$ be a stack in groupoids over $S$ and define $\mathcal{X}' = \mathcal{X} \times_S S'$. Suppose that $\mathcal{X}'$ is an algebraic stack over $S'$. Then $\mathcal{X}$ is an algebraic stack over $S$. If in addition $S' \to S$ is étale and $\mathcal{X}'$ is a Deligne–Mumford stack, then $\mathcal{X}$ is a Deligne–Mumford stack.*

Proof. Let us first prove item (1). If $U'$ is a scheme and $U' \to \mathcal{X}'$ a surjective and smooth morphism, then $U' \to \mathcal{X}' \to \mathcal{X}$ is surjective and smooth, and moreover étale if $\pi$ and $U' \to \mathcal{X}'$ are étale. Therefore, it suffices to prove that the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. For this, it suffices to consider to schemes $U$ and $V$, equipped with morphisms $U \to \mathcal{X}$ and $V \to \mathcal{X}$, and prove that the fibre product $U \times_{\mathcal{X}} V$ is representable by an algebraic space, see [LMB00, Corollary 3.13]. Define $U' = \mathcal{X}' \times_{\mathcal{X}} U$ and $V' = \mathcal{X}' \times_{\mathcal{X}} V$. We obtain the following cartesian diagram:




The morphism $\mathcal{X}' \to \mathcal{X}$ is representable, hence $U'$ and $V'$ are representable by algebraic spaces. Since $\mathcal{X}'$ is an algebraic stack, the morphism $V' \to \mathcal{X}'$ is representable by algebraic spaces, which implies that its base change $U' \times_{\mathcal{X}'} V' \to U'$ is representable by algebraic spaces. Finally, the morphism of algebraic spaces $U' \to U$ is étale and surjective, hence an epimorphism. Using [LMB00, Lemme 4.3.3], we conclude that the morphism $U \times_{\mathcal{X}} V \to U$ is representable. As $U$ is scheme, $U \times_{\mathcal{X}} V$ is an algebraic space, and we are done.

Next, we prove item (2). Via the composition $\mathcal{X}' \to S' \to S$, we may view $\mathcal{X}'$ as an algebraic stack over $S$, see [LMB00, Proposition 4.5]. In this way, we obtain a cartesian diagram of algebraic stacks over $S$:

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

As $S' \to S$ is representable, surjective and étale, the same holds for $\mathcal{X}' \to \mathcal{X}$. The stack $\mathcal{X}'$ is algebraic, hence $\mathcal{X}$ is algebraic as well, see item (1). □

Proof of Theorem 1.1. Proposition 3.2 yields the stack $\mathcal{X}$ over $S$ together with 1-isomorphism $\rho\colon \mathcal{X} \times_S S' \xrightarrow{\sim} \mathcal{X}'$ and the 2-isomorphism $\chi\colon p_2^*\rho \circ \mathrm{can} \Rightarrow \phi \circ p_1^*\rho$ that have the right compatibility properties with respect to $\psi$, so that we only need to prove that $\mathcal{X}$ is algebraic (resp. Deligne–Mumford if $S' \to S$ is surjective étale). This follows from Lemma 3.3. □

Even the case where $\mathcal{X}'$ is a scheme seems to yield a non-trivial result:

Corollary 3.4. *Let $S' \to S$ be a surjective étale morphism of schemes, and let $X'$ be a scheme over $S'$ equipped with a descent datum $\phi$ as in Section 2. Then there exists an algebraic space $X$ over $S$ and an $S$-morphism $\pi\colon X' \to X$ such that the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

*is cartesian. The tuple $(X, \pi\colon X' \to X)$ is compatible with the descent datum $\phi$ in an appropriate sense, and this makes $(X, \pi)$ unique up to isomorphism.*

Proof. Theorem 1.1 implies the existence of $X$ as a Deligne–Mumford stack, hence we only need to prove that $X$ is an algebraic space. For this, it suffices to show that the inertia group stack $I_X \to X$ is an equivalence, and hence to show that, for each scheme $T$ over $S$ and each object $x \in X(T)$, the map $Aut_X(x) \to T$ is an isomorphism of algebraic spaces, where $Aut_X(x)$ is the algebraic space over $T$ with $Aut_X(x)(T')$ the group of automorphisms of the object $x_{T'}$ that lie over the identity on $T'$. We may prove this locally; let $T' = T \times_S S'$ and $x' = x_{T'} \in X'(T')$. For each scheme $T''$ over $T'$, we have $Aut_X(x)_{T'}(T'') = Aut_{X'}(x')_{T'}(T'')$ which is trivial since $X'$ is a scheme. □

For a scheme $S$ and a stack $\mathcal{X}$, and a finite group $\Gamma$, a *group action of* $\Gamma$ *on* $\mathcal{X}$ *over* $S$ is an action of the functor in groups over $S$ associated to $\Gamma$ on the stack $\mathcal{X}$ over $S$, see [Rom05, Definition 1.3].

Lemma 3.5. *Let $S' \to S$ be a finite faithfully flat morphism of schemes which is a Galois covering with Galois group $\Gamma$, and let $\mathcal{X}'$ be a stack over $S'$. Then the following sets are in canonical bijection:*

*(1) The set of 2-descent data $(\phi, \psi)$ for $\mathcal{X}'$ over $S'$.*

(2) *The set of group actions of* $\Gamma$ *on* $\mathcal{X}'$ *as a stack over* $S$*, such that for each* $\sigma \in \Gamma$*, the composition* $\mathcal{X}' \xrightarrow{\sigma} \mathcal{X}' \to S'$ *agrees with the composition* $\mathcal{X}' \to S' \xrightarrow{\sigma} S'$*.*

(3) *The set of Galois* 2*-descent data for* $\mathcal{X}'$ *over* $S'$*.*

Proof. See [BLR90, Section 6.2, Example B] and [Poo17, Proposition 4.4.4] a the proof in the case of schemes. The stacky case is requires some straightforward generalizations; we leave the details to the reader. □

Proof of Theorem 1.3. See Theorem 1.1 and Lemma 3.5. □

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