MUMFORD'S CONSTRUCTION OF DEGENERATING ABELIAN VARIETIES

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1. Introduction

The goal of this talk is to discuss the paper [Mum72]. We fix:

Notation 1.1. Let A be an excellent normal noetherian ring with quotient field K. Let $I \subset A$ be an ideal with $I = \sqrt{I}$, and assume A is complete for the I-adic topology. For instance, A is a complete and normal noetherian local ring with maximal ideal I. We let $S = \mathrm{Spec}(A)$ with generic point $\eta \in S$, and define $S_0 = \mathrm{Spec}(A/I)$.

We start with two definition, fundamental in this talk:

Definition 1.2. Let $S = \mathrm{Spec}(A)$ as above.

(1) A *multiplicative torus* over S is a commutative group scheme over S which is étale locally isomorphic to a product of finitely many copies of $\mathbb{G}_m$.
(2) A *semi-abelian scheme* is a smooth, separated, finite type commutative group scheme $\pi\colon G \to S$ with geometrically connected fibres, such that each fibre G_s is an extension of an abelian variety A_s by a multiplicative torus T_s (that is, there is an exact sequence $0 \to T_s \to G_s \to A_s \to 0$).

Remark 1.3. The *torus rank* is the function $S \mapsto r(s) =$ (torus rank of G_s). This function is upper semi-continuous.

Fact 1.1. *Let G be a semi-abelian group schemes G of constant rank on S. Then G is globally over S an extension of an abelian scheme over S by a torus over S.*

Conjecture 1.4 (Mumford–Tate)**.** *Let G be a semi-abelian group scheme over S such that G_η is an abelian variety and G_0 has constant rank r over S_0. Then G is canonically represented as the 'quotient' of a semi-abelian group scheme $\widetilde{G}$ of constant rank r over the whole of S, by a discrete group $Y \subset \widetilde{G}(L)$ (L a finite extension of K) with $Y \cong \mathbb{Z}^r$.*

1.1. Intuition.

For $\tau \in \mathbb{H}$, consider the elliptic curve

$$G_\tau = \mathbb{C}/\left(\mathbb{Z} + \tau\mathbb{Z}\right) = \mathbb{C}^*/q^{\mathbb{Z}}, \qquad q = e^{2\pi i \tau} \in B^* = \{0 < |z| < 1\}.$$

Let $\mathcal{F}$ be the standard fundamental domain $\mathcal{F}$ for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Let $\mathcal{F}^0$ be $\mathcal{F}$ minus the points $t \in \mathbb{H}$ with $|t| = 1$ and the points with $|\Re(t)| = 1/2$. Let B be S times $\mathcal{F}^0$ plus the point $0 \in \mathbb{C}$. Then $S \cdot B^* = \mathcal{F}^0$, the interior of $\mathcal{F}$. Define an abelian scheme

$$G|_{B^*} = \left(\mathbb{C} \times B^*\right)/\mathbb{Z}^{\oplus 2} = \left(\mathbb{C}/\mathbb{Z} \times B^*\right)/\mathbb{Z} = \left(\mathbb{C}^* \times B^*\right)/\mathbb{Z}.$$

This extends to the semi-abelian scheme

$$G = (\mathbb{C} \times B)\,/\mathbb{Z}^{\oplus 2} = (\mathbb{C}/\mathbb{Z} \times B)\,/\mathbb{Z} = (\mathbb{C}^* \times B)\,/\mathbb{Z}.$$

Here, $\mathbb{Z}$ acts freely on $\mathbb{C}^* \times B^*$ by $n \cdot (x, \tau) = (e^{2\pi i \tau n} x, \tau) = (q(\tau)^n \cdot x, \tau)$. Notice that the torus rank is zero on B^* and one on $0 \in B$. Let

$$\widetilde{G} = (\mathbb{C} \times B)\,/\mathbb{Z} = \mathbb{C}/\mathbb{Z} \times B = \mathbb{C}^* \times B,$$

which is a semi-abelian group scheme of constant toric rank one on B. Then

$$\widetilde{G} \longrightarrow G$$

is a *global* uniformization of the semi-abelian scheme G by $\mathbb{C}^*$. It maps $\mathbb{C}^* \times \{0\}$ isomorphically onto $\mathbb{C}^* \times \{0\}$.

1.2. **Example.**

1.3. **General idea.** Mumford takes the approach to the other direction. The plan is as follows.

(1) Start with a split torus $\widetilde{G} \cong \mathbb{G}_m^r \times S$.
(2) Consider a set of period $Y \subset \widetilde{G}(K)$ satisfying suitable conditions.
(3) Construct a kind of compactification $\widetilde{G} \subset \widetilde{P}$ over S such that the action of Y by translation extends to $\widetilde{P}$.
(4) Take the I-adic completion $\widetilde{\mathfrak{P}}$ of $\widetilde{P}$, construct $\mathfrak{P} = \widetilde{\mathfrak{P}}/Y$, algebraize $\mathfrak{P}$ to a scheme P projective over S.
(5) Take a suitable open subset $G \subset P$.
(6) Prove that G is a semi-abelian group scheme over S that satisfies:

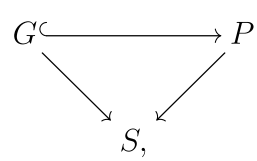
 Condition 1.5. *G_η is abelian and $G_0 \cong \widetilde{G}_0 = \mathbb{G}_m^r \times S_0$.*

(7) Prove that G is independent of the choice of $\widetilde{P}$.
(8) Prove that G uniquely determines Y.

We also have:

Theorem 1.6 (Mumford–Raynaud)**.** *If* $\dim(A) = 1$ *and A is local, then each semi-abelian group scheme G/S satisfying Condition* (1.5) *admits a uniformization.*

1.4. **Applications.** The construction takes as input a split torus $\widetilde{G} = \mathbb{G}_m^r \times S$ and outputs



where $G_\eta = P_\eta$ is an abelian variety over K, where $G_0 = \mathbb{G}_m^r \times S_0$ is a split torus over S_0, with compactification $G_0 \subset P_0$. Hence, this gives a *degeneration of abelian varieties* in two possible senses:

(1) Degenerate the abelian variety A_η to a group scheme which is no longer compact.

(2) Degenerate the abelian variety A_η to a proper variety which is no longer smooth.

We remark that the compactification $G \subset P$ is neither unique nor canonical. The fact that one has some choice in the construction has the advantage that, to quote Mumford, it allows one "to seek for the most elegant solutions in any particular case".

Remark 1.7. Let $X = \mathrm{Hom}(\widetilde{G}, \mathbb{G}_m)$ be the character group of $\widetilde{G}$. For $\alpha \in X$, let $\mathscr{X}^\alpha \in \mathcal{O}(\widetilde{G})^*$ be the corresponding section. Then for any $y \in \widetilde{G}(K)$ we get an element $\mathscr{X}^\alpha(y) \in K^*$. If $\dim(A) = 1$ then A is a complete discrete valuation ring. In that case, there is embedding $\widetilde{G}(K) \to \mathrm{Hom}(X, \mathbb{R})$ sending ξ to the map $\alpha \mapsto \mathrm{ord}\mathscr{X}^\alpha(\xi)$. Thus, as $Y \subset \widetilde{G}(K)$, we get a lattice $Y \subset \mathrm{Hom}(X, \mathbb{R})$. In particular, any polarization $\phi\colon Y \to X$ gives a quadratic form $Q_\phi\colon Y \to \mathbb{R}$. It is defined as

$$Q_\phi(y) = \mathrm{ord}\mathscr{X}^{\phi(y)}(y), \qquad y \in Y.$$

It extends to a positive definite quadratic form Q_ϕ on $E = \mathrm{Hom}(X, \mathbb{R})$.

2. The set-up

Let S be as in Notation ... Let $\widetilde{G} = \mathbb{G}_m^r \times S$ be a split torus of rank r over S. Let

$$X = \mathrm{Hom}_{\textit{Gr-Sch}/S}(\widetilde{G}, \mathbb{G}_{m,S}) \cong \mathrm{Hom}_{\textit{Gr-Sch}/S}(\mathbb{G}_{m,S}^r, \mathbb{G}_{m,S}) \cong \mathbb{Z}^r$$

be the character group of $\widetilde{G}$. For $\alpha \in X$, we let $\mathscr{X}^\alpha$ be the induced element in $\mathcal{O}(\widetilde{G})^*$. We have

$$\begin{aligned}\widetilde{G} &= \mathrm{Spec}A[\cdots, \mathscr{X}^\alpha, \cdots]_{\alpha\in X} / \left(\mathscr{X}^\alpha \cdot \mathscr{X}^\beta = \mathscr{X}^{\alpha+\beta}, \mathscr{X}^0 = 1\right) \\ &\cong \mathrm{Spec}A[x_1, \ldots, x_r, y_1, \ldots, y_r]/(x_i y_i = 1) \\ &\cong \mathrm{Spec}A[x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}].\end{aligned}$$

To see this, recall that

$$X \subset \mathrm{Hom}_{Sch/S}(\widetilde{G}, \mathbb{G}_{m,S}) = \mathrm{Hom}_{A-alg}(\mathcal{O}(\mathbb{G}_{m,S}), \mathcal{O}(\widetilde{G})) = \mathcal{O}(\widetilde{G})^*.$$

This gives a canonical element $\mathscr{X}^\alpha \in \mathcal{O}(\widetilde{G})$ for $\alpha \in X$. It is clear that the $\mathscr{X}^\alpha$ for $\alpha \in X$ satisfy the required relations; it remains to prove that they generate $\mathcal{O}(\widetilde{G})$ over A. For this, one may identify $\widetilde{G}$ with $\mathbb{G}_m^r \times S$, in which case it is obvious.

Notice that there is a natural pairing

$$\widetilde{G}(K) \times X \to K^*, \quad (y, \alpha) \mapsto \mathscr{X}^\alpha(y).$$

In fact, it is given as the composition

$$\widetilde{G}(K) \times X \to \widetilde{G}(K) \times \mathrm{Hom}(\widetilde{G}(K), \mathbb{G}_m(K)) \to \mathbb{G}_m(K)^* = K^*.$$

Definition 2.1. A set of *periods* is a subgroup $Y \subset \widetilde{G}(K)$ isomorphic to $\mathbb{Z}^r$.

Definition 2.2. Let Y be a set of periods. A *polarization* for the periods Y is a homomorphism

$$\phi\colon Y \to X$$

such that:

1. $\mathscr{X}^{\phi(y)}(z) = \mathscr{X}^{\phi(z)}(y)$ for all $y, z \in Y$,
2. $\mathscr{X}^{\phi(y)}(y) \in I$ for all $y \in Y$, $y \neq 0$.

By (2), the map ϕ is injective, because if $y \in Y$ is non-zero then $\mathscr{X}^{\phi(y)}(y) \neq 1$ hence $\phi(y) \neq 1$. In particular, $[X : \phi Y] < \infty$.

Definition 2.3. A *relatively complete model* of $\widetilde{G}$ with respect to the periods Y and the polarization ϕ consists of the following data:

1. an integral scheme $\widetilde{P}$, locally of finite type over A,
2. an open immersion $i\colon \widetilde{G} \to \widetilde{P}$,
3. an ample invertible sheaf $\widetilde{L}$ on $\widetilde{P}$,
4. an action of $\widetilde{G}$ on $\widetilde{P}$ and $\widetilde{L}$, denoted by

$$T_a\colon \widetilde{P} \to \widetilde{P}, \quad T_a^* L \to L, \quad a \in \widetilde{G}(S') \quad \text{for some } S\text{-scheme } S' \to S,$$

5. an action of Y on $\widetilde{P}$ and $\widetilde{L}$, denoted by

$$S_y\colon \widetilde{P} \to \widetilde{P}, \quad S_y^*\colon \widetilde{L} \to \widetilde{L}, \quad y \in Y;$$

these data must satisfy the following conditions:

(i) There exists an open $\widetilde{G}$-invariant subset $U \subset \widetilde{P}$ of finite type over S such that

$$\widetilde{P} = \bigcup_{y \in Y} S_y(U),$$

(ii) for all valuations v on $R(\widetilde{G})$, the field of rational functions on $\widetilde{G}$, for which $v \geq 0$ on A, v has a center on $\widetilde{P}$ if and only if

$$[\forall \alpha \in X \exists y \in Y \mid v(\mathscr{X}^{\alpha}(y) \cdot \mathscr{X}^{\alpha}) \geq 0]\,, \tag{1}$$

(iii) the action of $\widetilde{G}$ on $\widetilde{P}$ extends the action of $\widetilde{G}$ on itself by translation, and similarly, the action of Y on $\widetilde{P}_\eta$ extends the action of Y on $\widetilde{G}_\eta$ by translation ($Y \subset \widetilde{G}(K)$),

(iv) the actions of Y and $\widetilde{G}$ on $\widetilde{L}$ satisfy the identity

$$T_a^* S_y^* = \mathscr{X}^{\phi(y)}(a) \cdot S_y^* T_a^*$$

for all $y \in Y$ and all S'-valued points a of $\widetilde{G}$, for any S-scheme S'.

3. The existence of relative complete models

Let θ be an indeterminate and consider the graded ring

$$\mathscr{R} = \bigoplus_{k=0}^{\infty} \Big(\mathcal{O}(\widetilde{G}) \otimes_A K\Big) \cdot \theta^k = \Big(\mathcal{O}(\widetilde{G}) \otimes_A K\Big)[\theta] = \mathcal{O}(\widetilde{G}_\eta)[\theta].$$

Let Y act on $\mathscr{R}$ via operators S_y^*, $y \in Y$:

$$\begin{aligned} S_y^*(c) &= c, \quad c \in K, \\ S_y^*(\mathscr{X}^\alpha) &= \mathscr{X}^\alpha(y), \quad \alpha \in X, \\ S_y^*(\theta) &= \mathscr{X}^{\phi(y)}(y) \cdot \mathscr{X}^{2\phi(y)} \cdot \theta. \end{aligned}$$

Definition 3.1. A *star* Σ is a finite subset of X such that $0 \in \Sigma$, $-\Sigma = \Sigma$, and Σ contains a basis of X.

Definition 3.2. Let ϕ be a polarization and Σ a star. Let $R_{\phi,\Sigma}$ be the subring of $\mathscr{R}$ generated over A by the elements $S_y^*(\mathscr{X}^\alpha \theta)$ for $y \in Y, \alpha \in \Sigma$. That is,

$$R_{\phi,\Sigma} = A\left[\cdots, \mathscr{X}^{\phi(y)+\alpha}(y) \cdot \mathscr{X}^{2\phi(y)+\alpha} \cdot \theta, \cdots\right]_{y \in Y, \alpha \in \Sigma}.$$

We have:

Lemma 3.3. *If we replace ϕ by $n\phi$ for sufficiently large $n \in \mathbb{Z}$, then*

$$R_{\phi,\Sigma} \subset A[\cdots, \mathscr{X}^\alpha \cdot \theta, \cdots] = \mathcal{O}(\widetilde{G})[\theta].$$

Proof. It suffices to show $\mathscr{X}^{\phi(y)+\alpha}(y) \in A$ for all $y \in Y$ and $\alpha \in \Sigma$. □

Theorem 3.4. *Let $\widetilde{G}$ be a split torus over S, let $Y \subset \widetilde{G}(K)$ be a set of periods and let $\phi \colon Y \to X$ be a polarization. Then if ϕ is replaced by $n\phi$ for $n \in \mathbb{Z}_{\geq 0}$ sufficiently large, then* $\mathrm{Proj}(R_{\phi,\Sigma})$ *is a relatively complete model of $\widetilde{G}$ over S relative to Y and 2ϕ.*

Proof. We proceed in steps.

(1) Since $\mathrm{Proj}(R_{\phi,\Sigma})$ is the Proj of a graded ring generated by elements of degree one, it carries a canonical ample invertible sheaf $\mathcal{O}(1)$.

(2) The automorphisms S_y^* of $R_{\phi,\Sigma}$ induce automorphisms S_y of $\mathrm{Proj}(R_{\phi,\Sigma})$ and a compatible automorphism S_y^* of $\mathcal{O}(1)$.

(3) Define the action of $\widetilde{G}$ on $\mathrm{Proj}(R_{\phi,\Sigma})$ as follows. Let B be an A-algebra and let $a \in \widetilde{G}(B)$. We define an automorphism T_a^* of $B \otimes_A R_{\phi,\Sigma}$ as follows. Let

$$\begin{aligned} T_a^*(c) &= c, \quad c \in A \\ T_a^*(\mathscr{X}^\alpha) &= \mathscr{X}^\alpha(a) \cdot \mathscr{X}^\alpha, \quad \alpha \in X, \\ T_a^*(\theta) &= \theta. \end{aligned}$$

(4) Notice that $\mathrm{Proj}(R_{\phi,\Sigma})$ is covered by the affine open sets

$$U_{\alpha,y} = \mathrm{Spec} A\left[\cdots, \frac{\mathscr{X}^{\phi(z)+\beta}(z)}{\mathscr{X}^{\phi(y)+\alpha}(y)} \cdot \mathscr{X}^{2\phi(z-y)+\beta-\alpha}, \cdots\right]_{\beta \in \Sigma, z \in Y}.$$

There are only finitely many Y-orbits of the $U_{\alpha,y}$. The $\mathcal{O}(U_{\alpha,y})$ are integral domains, contained in $K(\cdots, \mathscr{X}^\alpha, \cdots) = R(\widetilde{G})$.

(5) We have

$$\begin{aligned} U_{0,0} &= \mathrm{Spec} A\left[\cdots, \mathscr{X}^{\phi(z)+\beta}(z)\cdot \mathscr{X}^{2\phi(z)+\beta}, \cdots\right]_{\beta\in\Sigma, z\in Y} \\ &= \mathrm{Spec} A\left[\cdots, \mathscr{X}^{\beta}, \cdots\right]_{\beta\in\Sigma} \\ &= \mathrm{Spec} A\left[\cdots, \mathscr{X}^{\beta}, \cdots\right]_{\beta\in X} = \widetilde{G}. \end{aligned}$$

(6) Show that $U_{\alpha,y}$ is of finite type over A.

(7) It follows that if $U = \cup_{\alpha\in\Sigma} U_{\alpha,0}$, then U is an open subset of $\mathrm{Proj}(R_{\phi,\Sigma})$ of finite type over A, such that

$$\bigcup_{y\in Y} S_y(U) = \mathrm{Proj}(R_{\phi,\Sigma}).$$

(8) Finally, check the completeness property for $\mathrm{Proj}(R_{\phi,\Sigma})$.

□

4. First properties of relative complete models

Notation 4.1. Let $\widetilde{G} \to S$ be a split torus of rank r as before, with S as in Notation ... Let $Y \subset \widetilde{G}(K)$ be a set of periods and let $\phi\colon Y \to X$ be a polarization. Moreover, let $\widetilde{P}$ be a relative complete model of $\widetilde{G}$ with respect to Y and ϕ.

Proposition 4.2. *We have $\widetilde{G}_\eta = \widetilde{P}_\eta$.*

Proposition 4.3. *Every irreducible component of $\widetilde{P}_0$ is proper over $S_0 = \mathrm{Spec}(A/I)$.*

Proof. Let Z be an irreducible component of $\widetilde{P}_0$. Let v be a valuation of $R(Z)$ with $v \geq 0$ on A/I. Let $V \subset R(Z)$ be the corresponding valuation ring. We first show that there exists $x \in Z$ with $\mathcal{O}_{Z,x} \subset V \subset R(Z)$. Let $z = \eta_Z$ be the generic point of Z. Choose a valuation ring V_1 with $\mathcal{O}_{\widetilde{P},z} \subset V_1 \subset R(\widetilde{P}) = R(\widetilde{G})$. Let $\mathfrak{p}_z \subset \mathcal{O}_{\widetilde{P},z}$ be the maximal ideal; then $\mathfrak{p}_z = \mathfrak{m} \cap \mathcal{O}_{\widetilde{P},z}$ for the maximal ideal $\mathfrak{m} \subset V_1$.

Let $V_2 \subset R(\widetilde{G})$ be the valuation ring of the composition v_2 of the valuations v and v_1. Then there is a prime ideal $\mathfrak{q} \subset V_2$ such that $V_{2,\mathfrak{q}} = V_1$ and $V_2/\mathfrak{q} = V$. One shows, using the completeness condition, that v_2 has a center on $\widetilde{P}$. Thus, there exists $x \in \widetilde{P}$ such that $\mathcal{O}_{\widetilde{P},x} \subset V_2$. The composition

$$\mathrm{Spec}(V_2) \to \mathrm{Spec}(V) \to Z \to \widetilde{P}$$

factors as $\mathrm{Spec}(V_2) \to \mathrm{Spec}(\mathcal{O}_{\widetilde{P},x}) \to \widetilde{P}$. Hence the closed point of $\mathrm{Spec}(V_2)$ is sent to $x \in \widetilde{P}$, which must therefore lie on Z. We get $x \in Z \subset \widetilde{P}$. Moreover, $\mathcal{O}_{\widetilde{P},x} \subset V_2$ implies $\mathcal{O}_{Z,x} \subset V$.

It remains to show that $\widetilde{P}_0$ is of finite type over S_0. Write $X = Z$ and $Y = S_0$. Let $f\colon X \to Y$ be the canonical map. To prove that f is of finite type, we may assume that $Y = \mathrm{Spec}(B)$ is integral and that f is dominant. Hence $B \subset R(X)$. Let $\mathcal{X}$ be the Zariski Riemann Surface of $R(X)$ over B. Set-theoretically, $\mathcal{X}$ is the set of valuations v on $R(X)$ which are ≥ 0 on B. For $x_1, \ldots, x_n \in R(X)$, we define $U(x_1, \ldots, x_n) \subset \mathcal{X}$ as the set of valuation rings $B \subset V \subset R(X)$ with $x_i \in V$ for all i. Then the $U(x_1, \ldots, x_n)$ form a basis of a topology on $\mathcal{X}$. Moreover, by what we have just shown, for each $v \in \mathcal{X}$ there is a unique $x \in X$ such that $\mathcal{O}_{X,x} \subset R_v \subset R(X)$. This defines a continuous map $\mathcal{X} \to X$. This map is surjective, as any $\mathcal{O}_{X,x}$ is contained in a valuation ring. As $\mathcal{X}$ is quasi-compact, the same holds for X. □

Corollary 4.4. *The closure $\bar{U}_0$ of U_0 in $\widetilde{P}_0$ is proper over S_0.* □

Proposition 4.5. *There is a finite subset $S \subset Y$ such that*

$$S_y(\bar{U}_0) \cap S_z(\bar{U}_0) = \emptyset$$

for all $y - z \notin S$. In other words, $\bar{U}_0 \cap S_y(\bar{U}_0) = \emptyset$ whenever $y \notin S$.

Corollary 4.6. *The group Y acts freely on $\widetilde{P}_0$.*

Proof. Let $x \in \widetilde{P}_0$ such that $S_y(x) \neq x$ for some $y \in Y$ with $y \neq 0$. There exists $z \in Y$ such that $S_z(x) \in U_0$. Moreover, $S_y S_z(x) = (S_z S_{y-z}) S_z(x) = S_z S_y(x) = S_z(x)$. Thus, we may assume that $x \in U_0$. But, as there is an infinite subgroup $G \subset Y$ that fixes x, this contradicts the proposition above. □

Theorem 4.7. *The scheme $\widetilde{P}_0$ is connected.*

Proof. Since A is complete in the I-adic topology and has no non-trivial idempotents, then A/I has no non-trivial idempotents either. Indeed, if $e^2 = e + y$ for some $e \in A$ and $y \in I$, then, as the topological spaces $\mathrm{Spec}(A/I)$ and $\mathrm{Spec}(A/I^n)$ are the same for all $n \geq 1$, we get a compatible system of idempotents $e_n \in A/I^n$. Thus, we get an idempotent in $A = \varprojlim A/I^n$, which must be one.

So S_0 is connected. Thus, $\widetilde{G}_0$ is a connected open subset of $\widetilde{P}_0$, and the claim is that $\widetilde{G}_0$ is dense in $\widetilde{P}_0$. □

5. Construction of the quotient

Theorem 5.1. *For every $n \geq 1$, there exists a scheme P_n, projective over $S_n = \mathrm{Spec}(A/I^n)$, an ample sheaf $\mathcal{O}(1)$ on P_n, an an étale surjective morphism*

$$\pi\colon \widetilde{P} \times_S S_n \longrightarrow P_n,$$

such that set-theoretically, $\pi(x) = \pi(y)$ if and only if x and y are in the same Y-orbit, and such that $\mathcal{O}(1)$ on $\widetilde{P} \times_S S_n$ is the pull-back of $\mathcal{O}(1)$ on P_n.

Proof. First, let $k \geq 1$ be an integer such that under the action of $kY \subset Y$, no two points of any open sset

$$S_y(U) \times_S S_n \subset \widetilde{P} \times_S S_n$$

are identified. Then we can form the quotient:

$$\pi'\colon \widetilde{P} \times_S S_n \longrightarrow P'_n$$

of $\widetilde{P} \times_S S_n$ by the group kY. It is easily checked that P'_n is proper over S_n, and that $\mathcal{O}(1)$ descends to an ample line bundle $\mathcal{O}(1)$ on P'_n. Thus, P'_n is projective over S_n.

Then the finite group Y/kY acts freely on the projective scheme P'_n and on the ample sheaf $\mathcal{O}(1)$, so the quotient $P_n = P'_n/(Y/kY)$ exists, and $\mathcal{O}(1)$ descends to an ample line bundle $\mathcal{O}(1)$ on P_n. □

Now, define a formal scheme as follows:

$$\mathfrak{P} = \varinjlim P_n.$$

The sheaves $\mathcal{O}(1)$ fit together to form an ample sheaf $\mathcal{L}$ on $\mathfrak{P}$.

Proposition 5.2. *There exists a unique scheme P, proper over S, such that $\widehat{P} = \varinjlim P \times_S S_n = \mathfrak{P}$ over $\widehat{S}$. Moreover, $\mathcal{L} = \widehat{L}$ for an S-relatively ample line bundle L on P.*

Proof. This follows by applying Grothendieck's algebraization theorem. □

Next, define

$$\begin{aligned} G_n &= \bigcup_{y\in Y} \left(S_y(\widetilde{G}) \times_S S_n\right)/Y \subset P_n, \\ \varinjlim G_n &= \mathfrak{G} \subset \mathfrak{P}, \\ \widetilde{B} &= \widetilde{P} - \bigcup_{y\in Y} S_y(\widetilde{G}) \subset \widetilde{P}, \\ B_n &= \left(\widetilde{B} \times_S S_n\right)/Y \subset P_n, \\ \varinjlim B_n &= \mathfrak{B} \subset \mathfrak{P}. \end{aligned}$$

Then $\mathfrak{B} = \widehat{B}$ for a reduced closed subscheme $B \subset P$. Define

$$G = P - B.$$

Lemma 5.3. *We have $\widehat{G} = \widehat{\widetilde{G}}$ as formal schemes over $\widehat{S} = \mathrm{Spf}(A)$.*

Proof. Indeed, we have on the one hand, that

$$\widehat{G} = \widehat{P} - \widehat{B} = \mathfrak{P} - \mathfrak{B} = \varinjlim(P_n - B_n) = \varinjlim G_n = \mathfrak{G}.$$

On the other hand, $G_n \cong \widetilde{G} \times_S S_n$, so that

$$\mathfrak{G} = \varinjlim G_n = \varinjlim \widetilde{G} \times_S S_n = \widehat{\widetilde{G}}.$$

This proves the lemma. □

6. Properties of the quotient

Proposition 6.1. *We have:*

1. *G is smooth over S.*
2. *P is irreducible.*

Theorem 6.2. *Let $(\widetilde{G}_i, Y_i, \phi_i, \widetilde{P}_i)$ be two tori plus periods, polarizations and relatively complete models ($i = 1, 2$). Let G_i be the two schemes constructed as above. Then, for all S-homomorphisms*

$$\widetilde{f}\colon \widetilde{G}_1 \to \widetilde{G}_2$$

such that $\widetilde{f}(Y_1) \subset Y_2$, there exists a unique S-morphism

$$f\colon G_1 \to G_2$$

such that, under the canonical isomorphisms $\widehat{G}_i = \widehat{\widetilde{G}}_i$, we have that f and $\widetilde{f}$ are formally identical.

Corollary 6.3. *The scheme G depends only on the torus $\widetilde{G}$ and the periods Y, and is independent of the polarization ϕ and the relatively complete model $\widetilde{P}$.* □

Corollary 6.4. *The S-scheme G is a naturally group scheme over S.*

Proof. Apply the theorem to $(\widetilde{G} \times_S \widetilde{G}, Y \times Y, \phi \times \phi, \widetilde{P} \times_S \widehat{P})$ and $(\widetilde{G}, Y, \phi, \widetilde{P})$. This gives $\mu\colon G \times_S G \to G$. Similarly, get $i\colon G \to G$. □

Corollary 6.5. *G_η is an abelian variety.*

Proof. We have $\widetilde{G}_\eta = \widetilde{P}_\eta$. Hence $\widetilde{B}_\eta = \emptyset$. Thus $\mathcal{O}_{\widetilde{B}} \otimes_A A_\tau = 0$ for some $\tau \in A$, non-zero. This implies that $\tau^n \cdot \mathcal{O}_B = 0$ for some n. Hence $B_\eta = \emptyset$. Hence $G_\eta = P_\eta$ is proper over K. Since G is irreducible, G_η is irreducible. Hence, it is an abelian variety. □

Proposition 6.6. *G_s is connected for $s \in S$, and G is a semi-abelian scheme over S.*

References

[Mum72] David Mumford. "An analytic construction of degenerating abelian varieties over complete rings". In: *Compositio Mathematica* 24 (1972), pp. 239–272.