Contractions of K_X-negative extremal rays : Three examples and a flip

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These notes are meant to work out the details of some examples provided by Debarre in his book [Deb01]. In these notes, all schemes are defined over an algebraically closed field k.

Let X be a scheme and let $\mathscr{E}$ be a locally free sheaf on X. Consider the contravariant functor

$$F \colon \mathsf{Sch}/X \to \mathsf{Set}, \quad (\pi \colon T \to X) \mapsto \{(\mathcal{L} \in \operatorname{Pic}(T), f \colon \pi^*\mathscr{E} \twoheadrightarrow \mathcal{L})\}/\cong .$$

Then F is representable by an X-scheme $\pi \colon \mathbb{P}(\mathscr{E}) \to X$ [Gro61, II, Proposition 4.3.2]. Since

$$\operatorname{Hom}(\mathbb{P}(\mathscr{E}), \mathbb{P}(\mathscr{E})) = F(\mathbb{P}(\mathscr{E})) = \{\pi^*\mathscr{E} \twoheadrightarrow \mathcal{L}\}/\cong,$$

the identity $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E})$ gives rise to a quotient $Q \colon \pi^*\mathscr{E} \to \mathcal{O}_X(1)$, well-defined up to isomorphism, and the tuple $((\mathcal{O}_X(1), Q)$ is universal in the sense that for any $T \to X$, line bundle $\mathcal{L}$ on T and $\alpha \colon \mathscr{E}_T \twoheadrightarrow \mathcal{L}$, there is a unique $f \colon T \to \mathbb{P}(\mathscr{E})$ over X such that $f^*(Q) \cong \alpha$.

1 Example I: · a fiber contraction

Let $\mathscr{E}$ be a vector bundle of rank $r+1$ on a smooth projective variety Y and let $X = \mathbb{P}(\mathscr{E})$, the bundle of hyperplanes in the fibers of $\operatorname{Spec}(\operatorname{Sym}(\mathscr{E})) \to X$. Now let $t \in Y(k)$, and let $L \hookrightarrow \mathbb{P}(\mathscr{E})_t = \mathbb{P}(\mathscr{E}_t)$ be a line in the projective space $\mathbb{P}(\mathscr{E}_t)$ over k. Then $L \hookrightarrow X_t \hookrightarrow X$ is a curve in X. Let $\ell \in \mathrm{N}_1(X)$ be its class.

Proposition 1.1. *The ray $R := \mathbb{R}^+ \cdot \ell \subset \mathrm{NE}(X)$ is K_X-negative and extremal. The morphism*

$$\pi \colon X = \mathbb{P}(\mathscr{E}) \to Y$$

is the contraction of R, and π is a fiber contraction: *X is covered by curves contracted by π.*

To prove this, we need two lemmata.

Lemma 1.2. *Let $\xi \in \mathrm{N}^1(X)$ be the class of the line bundle $\mathcal{O}_X(1)$, the universal quotient of $\mathscr{E}$. Then*

$$K_X = -(r+1)\xi + \pi^*(K_Y + \det(\mathscr{E})) \in \mathrm{N}^1(X).$$

Proof. Since the morphism $X \to Y$ is smooth, the following sequence of $\mathcal{O}_X$-modules

$$0 \to \pi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0 \tag{1}$$

is exact (see [Stacks, Tag 02K4]). From (1), we get that

$$K_X = \det(\Omega_X) = \pi^*(K_Y) \otimes \det(\Omega_{X/Y}).$$

On the other hand, the (generalized) Euler sequence is an exact sequence

$$0 \to \Omega_{X/Y} \to \pi^*\mathscr{E} \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0, \tag{2}$$

which gives

$$\pi^* \det(\mathscr{E}) \otimes \mathcal{O}_X(-(r+1)) = \det(\Omega_{X/Y}).$$

Therefore,

$$K_X = \pi^* K_Y + \pi^* \det(\mathscr{E}) - (r+1) \cdot \mathcal{O}_X(1) = -(r+1)\xi + \pi^*(K_Y + \det(\mathscr{E})).$$

□

Corollary 1.3. *One has*

$$K_X \cdot \ell = -(r+1).$$

Proof. Indeed, if $j\colon X_t \to X$ is the inclusion of X_t in X, then $\ell = j_*([L])$ and since the composition $X_t \to X \to Y$ factors over $\operatorname{Spec}(k(t)) \hookrightarrow Y$, we have

$$K_X \cdot \ell = K_X \cdot j_*[L] = j^*(K_X) \cdot [L] = j^*(-(r+1)\xi) \cdot [L] = \mathcal{O}_{\mathbb{P}(\mathscr{E}_t)}(-r-1) \cdot [L] = -r-1.$$

□

Lemma 1.4. *The class ℓ spans a K_X-negative ray $R \subset \operatorname{NE}(X)$ whose contraction is $\pi\colon X \to Y$.*

Proof. By Corollary 1.3, the class ℓ is K_X-negative. By definition,

$$\operatorname{NE}(\pi) = \operatorname{NE}(X) \cap \operatorname{Ker}\left(\pi_*\colon \operatorname{NE}(X) \to \operatorname{NE}(Y)\right).$$

So $R \subset \operatorname{NE}(\pi)$. To prove that $\operatorname{NE}(\pi) = R$, we must show that if an irreducible curve $C \subset X$ is contracted by π, then $[C]$ is a multiple of ℓ. But necessarily, such a curve $C \subset X$ is contained in a closed fiber $\mathbb{P}(\mathscr{E}_b) = X_b \subset X$, $b \in Y(k)$. Hence $[C] = \iota_*([C'])$ for a curve $C' \subset X_b$, where $\iota\colon X_b \to X$ is the inclusion. Since

$$\operatorname{CH}_1(\mathbb{P}(\mathscr{E}_b)) = \mathbb{Z} \cdot [L_b]$$

for a line $L_b \subset \mathbb{P}(\mathscr{E}_b))$, we have $[C'] = n \cdot [L_b] \in \mathrm{N}_1(X_b)$ for some $n \in \mathbb{Z}_{\geq 1}$ (the degree of C').

We claim that $\iota_*[L_b] = j_*[L] = \ell \in \mathrm{N}_1(X)$. Indeed, denote $\mathbb{K} = \mathcal{O}_X(1)^{r-1} \in \operatorname{CH}^{r-1}(X)$ and choose a smooth irreducible variety $\mathscr{C} \in \mathbb{K}$. Then $\mathbb{K}_y = \mathcal{O}_{\mathbb{P}(\mathscr{E})_y}(1)^{r-1}$ for each $y \in Y$, and in particular $\mathbb{K}_t = \ell$ and $\mathbb{K}_b = [L_b]$. We have $\mathscr{C} \subset X \to Y$, a family of curves in the fibers of π such that $\mathscr{C}_y$ is a line in $\mathbb{P}(\mathscr{E})_y$ for each $y \in Y(k)$. Now we simply note that b and t are algebraically equivalent on Y, since $\deg(b) = \deg(t) = 1$, and that $\pi\colon \mathscr{C} \to Y$ is flat. Consequently, C_b and C_t are algebraically equivalent in $\mathscr{C}$, hence their pushforwards to X are algebraically equivalent in X.

Therefore,

$$[C] = \iota_*[C'] = \iota_*(n \cdot [L_b]) = n \cdot j_*[L] = n \cdot \ell.$$

□

Proof of Proposition 1.1. Because $R = \operatorname{NE}(\pi)$, the ray $R \subset \operatorname{NE}(X)$ is extremal [Deb01, Proposition 1.14]. Moreover, $\pi\colon X \to Y$ is a *fiber contraction*, X is *uniruled*, the image of π has dimension less than X and the general fiber of π is a *Fano variety* (see [Deb01, §7.42] for why the latter is true in general).

To see why π is a fiber contraction, let $L(R) \subset X$ be the locus of R; we must show that $L(R) = X$. But this is clear: for each $y \in Y(k)$, the fiber $X_y \subset X$ is a projective space, hence covered by lines; the above shows that all these lines give the same class $\ell \in \mathrm{N}_1(X)$. Hence X is covered by curves $C \subset X$ whose class $[C] \in \operatorname{NE}(X)$ lies in $R = \mathbb{R}^+ \cdot \ell$. □

2 Example II: · a divisorial contraction

Now let Y be a smooth projective variety, let $Z \subset Y$ be a smooth subvariety of codimension $c \geq 2$, and let $\pi\colon X \to Y$ be the blow-up of Z. Let $E \subset X$ be the exceptional divisor. By Lemma 2.2 below, we have

$$K_X = \pi^*(K_Y) + (c-1)E \quad \in \quad \mathrm{N}^1(X). \tag{3}$$

Let $t \in Z(k)$, and $F = \pi^{-1}(t) \subset X$. Then let $L \subset F$ be a line contained in $F \cong \mathbb{P}^{c-1}$. Finally, let $\ell = [L] \in \mathrm{NE}(X)$.

Proposition 2.1. *1. The ray $R := \mathbb{R}^+ \cdot \ell \subset \mathrm{NE}(X)$ is K_X-negative.*

2. The morphism $\pi\colon X \to Y$ is the contraction of R.

3. The morphism π is a divisorial contraction*: the union $L(R) \subset X$ of curves $C \subset X$ contracted by π is a divisor (in fact an irreducible divisor by [Deb01, Proposition 6.10]).*

Proof. 1. Let $j\colon F \hookrightarrow E \hookrightarrow X$ be the inclusion of F in X. We have $K_X \cdot \ell = (c-1)E \cdot j_*[L] = j^*(K_X) \cdot [L] = j^*\mathcal{O}_X((c-1)E) \cdot [L] = \mathcal{O}_F((c-1)E) \cdot [L] = (c-1) \cdot \mathcal{O}_F(E) \cdot [L]$. But $\mathcal{O}_E(E) = \mathscr{N}_{E/X} = \mathcal{O}_E(-1)$, hence $\mathcal{O}_F(E) = \mathcal{O}_F(-1)$. Therefore, $K_X \cdot \ell = -(c-1)$.

2. Let $C \subset X$ be a curve contracted by π. Then $C \subset E_z \subset E$ for some $z \in Z$. Therefore $[C] = m \cdot [L_z] \in \mathrm{NE}(E_z)$. Consider $\mathcal{O}_E(1)^{c-2}$ and let $\mathscr{C} \in \mathcal{O}_E(1)^{c-2} \in \mathrm{CH}^{c-2}(E)$ be a smooth irreducible variety. Then $\mathscr{C} \subset E \to Z$ is a family of curves in the fibers of $E \to Z$. Since z and t are algebraically equivalent on z, we have $[L_z] = [L] \in \mathrm{NE}(E)$. Consequently, $[C] = m \cdot \ell \in \mathrm{NE}(X)$. Hence a curve is contracted by π if and only if it is numerically equivant to a multiple of ℓ. This proves that $\mathrm{NE}(\pi) = \mathrm{Ker}(\pi_*) \cap \mathrm{NE}(X) = R$.

3. It is clear from the above that $L(R) = E$.

□

2.1 Divisors on blow-ups

Let X be a smooth projective variety, let Y be a smooth subvariety of Y of codimension r, and let $\pi\colon \tilde{X} \to X$ be the blow-up of X along Y. Let $Y' = \pi^{-1}(Y)$.

Lemma 2.2. *1. The maps $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\tilde{X})$ defined by $n \mapsto [nY']$ give rise to an isomorphism $\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$.*

2. We have $K_{\tilde{X}} \cong \pi^*(K_X) \otimes \mathcal{O}_{\tilde{X}}((r-1)Y')$.

Proof. 1. We know that if $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Y, then the morphism $f\colon Y' \to Y$ corresponds to the projection

$$Y' = \mathbb{P}(\mathcal{I}/\mathcal{I}^2) \xrightarrow{f} Y.$$

Let $U = \tilde{X} \setminus Y' \subset \tilde{X}$, and let $V = X \setminus Y$. Then $\pi|_U\colon U \to V$ is an isomorphism. There is an exact sequence

$$\mathrm{CH}_{n-1}(Y') \to \mathrm{Pic}(\tilde{X}) \to \mathrm{Pic}(U) \to 0.$$

This fits in the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Pic}(X) & \xrightarrow{\sim} & \mathrm{Pic}(V) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi^*} & & {\scriptstyle \sim}\downarrow{\scriptstyle \pi^*} & & \\
0 & \longrightarrow & \mathrm{CH}^0(Y) & \longrightarrow & \mathrm{Pic}(\tilde{X}) & \longrightarrow & \mathrm{Pic}(U) & \longrightarrow & 0.
\end{array}$$

2. Write
$$K_{\tilde{X}} \cong \pi^*(\mathscr{M}) \otimes \mathcal{O}_{\tilde{X}}(mY')$$
for some invertible sheaf $\mathscr{M}$ on X and some integer m. Then
$$K_U \cong K_{\tilde{X}}|_U \cong \pi^*(\mathscr{M}|_V).$$
Since $\pi\colon U \to V$ is an isomorphism, we have $\mathscr{M}|_V \cong K_V$. But $\mathrm{Pic}(X) \to \mathrm{Pic}(V)$ is also an isomorphism, so $\mathscr{M} \cong K_X$. Consequently, we obtain
$$K_{\tilde{X}}(Y') \cong \pi^*(K_X) \otimes \mathcal{O}_{\tilde{X}}((m+1)Y').$$
Therefore,
$$K_{Y'} \cong K_{\tilde{X}}(Y')|_{Y'} \cong \pi^*(K_X)|_{Y'} \otimes \mathcal{O}_{Y'}(Y')^{\otimes(m+1)} \cong \pi^*(K_X)|_{Y'} \otimes \mathscr{N}_{Y'/\tilde{X}}^{\otimes(m+1)}.$$
But $\mathscr{N}_{Y'/\tilde{X}} \cong \mathcal{O}_{Y'}(-1)$ under the identification $Y' \cong \mathbb{P}(\mathcal{I}/\mathcal{I}^2) \to Y$, so that
$$K_{Y'} \cong \pi^*(K_X)|_{Y'} \otimes \mathcal{O}_{Y'}(-m-1).$$
Let $t \in Y$ be a closed point, and let $Z \subset Y'$ be the fibre of Y' over t. Then the codimension of Z in Y' equals $n-r$, where $n = \dim(X) = \dim(\tilde{X})$. Hence by the adjunction formula,
$$K_Z \cong K_{Y'}|_Z \otimes \bigwedge^{n-r} \mathscr{N}_{Z/Y'} \cong \mathcal{O}_Z(-m-1) \otimes \det(\mathscr{N}_{Z/Y'}).$$
The following lemma shows that $\det(\mathscr{N}_{Z/Y'}) \cong \mathcal{O}_Z$:

Lemma 2.3. *Let $\phi\colon \mathscr{X} \to B$ be a smooth projective morphism over a smooth variety B. Let $0 \in B(k)$ and $X = \phi^{-1}(0) \subset \mathscr{X}$. Then*
$$\mathscr{N}_{X/\mathscr{X}} \cong \phi^*(T_B)|_X \cong T_{B,0} \otimes_k \mathcal{O}_X.$$

Proof. It sufficest to verify the exactness of the sequence
$$0 \to T_X \to T_{\mathscr{X}}|_X \to \phi^*(T_B)|_X \to 0.$$
So let $x \in X(k)$; we need to show that the sequence
$$0 \to T_{X,x} \to T_{\mathscr{X},x} \to T_{B,0} \to 0$$
is exact. This follows from the fibre product
$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & B, \end{array}$$
for a tangent vector $v\colon \mathrm{Spec}(k)[\varepsilon] \to \mathscr{X}$ with $v(\mathrm{Spec} k) = x$ factors through $0\colon \mathrm{Spec}(k)[\varepsilon] \to B$ if and only if v lifts as a tangent vector $w\colon \mathrm{Spec}(k)[\varepsilon] \to X$ with $w(\mathrm{Spec}(k)) = x$. □

On the other hand, $Z \cong \mathbb{P}^{r-1}$, so that $K_Z \cong \mathcal{O}_Z(-r)$. Therefore, $r = m+1$, i.e. $m = r-1$, and we conclude that
$$K_{\tilde{X}} \cong \pi^*(K_X) \otimes \mathcal{O}_{\tilde{X}}((r-1)Y')$$
as desired.

□

3 Example III: · a small contraction

3.1 The set-up

1. Let r and s be positive integers, and let $\mathscr{E} = \mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(r+1)}$. We define $Y = Y_{r,s} = \mathbb{P}(\mathscr{E})$, and let
$$\pi \colon Y = \mathbb{P}(\mathscr{E}) \to \mathbb{P}^s$$
be the canonical morphism. The quotient $\mathscr{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^s}$ corresponds to a section $\sigma \colon \mathbb{P}^s \to Y$ of π; we define $P = \sigma(\mathbb{P}^s) \subset Y$. We then let
$$\varepsilon_1 \colon X_{r,s} \to Y_{r,s}$$
be the blow-up of P in Y. In a similar way, define $\varepsilon_2 \colon X_{s,r} \to Y_{s,r}$.

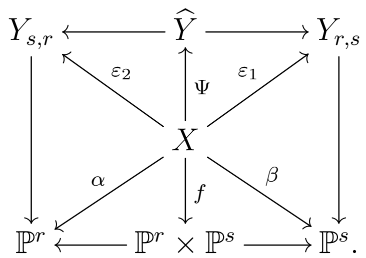
2. Let $\ell \in \mathrm{N}^1(Y_{r,s})$ be the class of a line $L \subset P$, and let $\ell' \in \mathrm{N}^1(Y_{r,s})$ be the class of a line $\Gamma \subset Z$ in a fiber $Z \subset Y_{r,s}$ of $\pi \colon Y_{r,s} \to \mathbb{P}^s$.

3. Let $\ell \in \mathrm{NE}(X_{r,s})$ be the class of a line $l \subset E$ that projects onto $L \subset P \subset Y_{r,s}$, but is contracted by the projection $E \cong \mathbb{P}^s \times \mathbb{P}^r \to \mathbb{P}^r$ (see Lemma 3.5 for the fact that $E \cong \mathbb{P}^s \times \mathbb{P}^r$). Let ℓ'' be the class of a line in E contracted by ε, and ℓ' the class of the strict transform of a line in a fiber of $\pi \colon Y_{r,s} \to \mathbb{P}^s$ that meets $P \subset Y_{r,s}$.

Lemma 3.1. *Let $D \subset Y_{r,s}$ be the divisor, isomorphic to $\mathbb{P}^r \times \mathbb{P}^s$, corresponding to the quotient $\mathscr{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^s}(1)^{r+1}$. Then D is contained in the base-point-free linear system $|\mathcal{O}_{Y_{r,s}}(1)|$. Let*
$$c_{r,s} \colon Y_{r,s} \to \mathbb{P}^{(r+1)(s+1)}$$
be the induced morphism. Then the image $\widehat{Y}_{r,s}$ of $c_{r,s}$ is the cone over the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^{(r+1)(s+1)}$ [Gro61, II §4.3]. In particular, $\widehat{Y}_{r,s} = \widehat{Y}_{s,r} = \widehat{Y}$.

Theorem 3.2.

- *There is a canonical isomorphism $X_{r,s} \cong X_{s,r}$. Thus, if we define $X = X_{r,s}$, we obtain a commutative diagram*



- *We have $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X) = \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell' + \mathbb{R}^+ \cdot \ell''$. We have the following equalities:*
  - $\mathrm{NE}(\varepsilon_2) = \mathbb{R}^+ \cdot \ell$
  - $\mathrm{NE}(f) = \mathbb{R}^+ \cdot \ell'$
  - $\mathrm{NE}(\varepsilon_1) = \mathbb{R}^+ \cdot \ell''$
  - $\mathrm{NE}(\alpha) = \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell'$
  - $\mathrm{NE}(\beta) = \mathbb{R}^+ \cdot \ell' + \mathbb{R}^+ \cdot \ell''$.
  - $\mathrm{NE}(\Psi) = \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell''$.

Remarks 3.3. • The corresponding *flip* is the composition

$$Y_{r,s} \dashrightarrow X \longrightarrow Y_{s,r}.$$

- It is an isomorphism in codimension one (in fact, it is an isomorphism outside the subvarieties $P_{r,s} \subset Y_{r,s}$ and $P_{s,r} \subset Y_{s,r}$).
- The above remark implies that the Picard numbers of $Y_{r,s}$ and $Y_{s,r}$ are the same.
- The $K_{Y_{r,s}}$-negative extremal ray $R = \mathbb{R}^+ \cdot \ell$ on $Y_{r,s}$ becomes a $K_{Y_{s,r}}$-*positive* extremal ray $R = \mathbb{R}^+ \cdot \ell \subset \mathrm{NE}(Y_{r,s})$ (see Lemma 4.4).

The rest of this section is devoted to the proof of Theorem 3.2.

3.2 The cone of curves in Y

The homomorphism $\pi^*\colon \mathrm{Pic}(\mathbb{P}^s) \to \mathrm{Pic}(Y)$ induces an isomorphism

$$\mathrm{Pic}(Y) = \mathrm{Pic}(\mathbb{P}^s) \oplus \mathbb{Z} \cdot [\mathcal{O}_Y(1)], \tag{4}$$

see [Har77, Exercice II.7.9]. In particular, if $\xi = [\mathcal{O}_Y(1)] \in \mathrm{N}^1(Y)$ and $h = [\pi^*(\mathcal{O}_{\mathbb{P}^s}(1))]$, then

$$\mathrm{N}^1(Y) = \mathbb{Z} \cdot \xi \oplus \mathbb{Z} \cdot h.$$

Lemma 3.4. *We have*

$$\mathrm{NE}(Y) = \overline{\mathrm{NE}}(Y) = \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell'.$$

Proof. The composition $P \to Y \to \mathbb{P}^s$ is an isomorphism; let $q \in \mathrm{N}^1(\mathbb{P}^s)$ be the class of the line in $\mathbb{P}^s$ corresponding to ℓ. Let $j\colon P \to Y$ and $\iota\colon Z \to Y$ be the inclusions.

First observe that $j^*(\xi) = 0$. Indeed, $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}) \subset \mathbb{P}(\mathscr{E})$ is the moduli space of invertible quotients of $\mathcal{O}_{\mathbb{P}^s}$, which are necessarily trivial. In other words: if $\phi\colon P \to Y \to \mathbb{P}^s$ is the composition, then $\phi^*(\mathcal{O}_Y(1))$ is the quotient $\phi^*(\mathcal{O}_{\mathbb{P}^s}) = \mathcal{O}_P$ of $\phi^*(\mathscr{E})$ by construction of P.

We have

$$h \cdot \ell = \pi^*\mathcal{O}(1) \cdot \ell = \pi^*\mathcal{O}(1) \cdot j_*[L] = \mathcal{O}(1) \cdot \pi_*(j_*[L]) = \mathcal{O}(1) \cdot q = 1, \tag{5}$$
$$h \cdot \ell' = \pi^*\mathcal{O}(1) \cdot \ell' = \pi^*\mathcal{O}(1) \cdot \iota_*[\Gamma] = \mathcal{O}(1) \cdot \pi_*\iota_*[\Gamma] = 0, \tag{6}$$
$$\xi \cdot \ell = \mathcal{O}_Y(1) \cdot j_*[L] = j^*\mathcal{O}_Y(1) \cdot [L] = 0 \cdot [L] = 0, \tag{7}$$
$$\xi \cdot \ell' = \mathcal{O}_Y(1) \cdot \iota_*\Gamma = \iota^*\mathcal{O}_Y(1) \cdot \Gamma = \mathcal{O}_Z(1) \cdot \Gamma = 1. \tag{8}$$

Let $C \subset Y$ be an irreducible curve. We may write

$$C = a \cdot \ell + b \cdot \ell' \in \mathrm{NE}(Y).$$

The above equations show that

$$C \cdot h = a, \quad C \cdot \xi = b.$$

We claim that ξ and h are nef. Indeed, ξ is ample and h is the pull-back of a nef line bundle, hence nef (projection formula).

Therefore, $C \cdot h = a \geq 0$ and similarly, $b \geq 0$. This proves that $[C] = a \cdot \ell + b \cdot \ell' \in \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell'$. Since C was arbitrary, this proves that $\mathrm{NE}(Y) \subset \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell'$. Consequently,

$$\mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell' \subset \mathrm{NE}(Y) \subset \overline{\mathrm{NE}}(Y) \subset \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell'$$

because the cone $\mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell' \subset \mathrm{N}_1(Y)_{\mathbb{R}}$ is closed. □

3.3 The multiplication table of X

Lemma 3.5. *1. The normal bundle $\mathscr{N}_{P/Y}$ is isomorphic to $\mathcal{O}_P(-1)^{\oplus(r+1)}$.*

2. The exceptional divisor $E \subset X$ is isomorphic to $\mathbb{P}^r \times \mathbb{P}^s$.

3. The line bundle $\mathcal{O}_E(E)$ is of type $(-1,-1)$.

Proof. 1.

2. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal sheaf of P. We know [Har77, Theorem II.8.24] that

$$E = \mathbb{P}(\mathcal{I}/\mathcal{I}^2) = \mathbb{P}(\mathscr{N}_{P/Y}^{\vee}) = \mathbb{P}(\mathcal{O}_P(1)^{\oplus(r+1)}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(r+1)}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^{\oplus(r+1)}) \cong \mathbb{P}^r \oplus \mathbb{P}^s.$$

3. Define $W = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(r+1)})$. Then under the above isomorphism, we have

$$\mathscr{N}_{E/X} \cong \mathcal{O}_W(-1).$$

Moreover, let $\pi' \colon V = \mathbb{P}^r \oplus \mathbb{P}^s = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^{\oplus(r+1)}) \to \mathbb{P}^s$ be the projection. Then under the isomorphism

$$\varphi \colon \mathbb{P}(\mathcal{O}_{\mathbb{P}^s}^{\oplus(r+1)}) = V \xrightarrow{\sim} W,$$

we have (see [Gro61, II, Proposition 4.1.4] or [Har77, Lemma II.7.9]):

$$\pi_r^*(\mathcal{O}_{\mathbb{P}^r}(1)) = \mathcal{O}_V(1) \cong \varphi^*\mathcal{O}_W(1) \otimes (\pi')^*\mathcal{O}_{\mathbb{P}^s}(-1) \quad \Longrightarrow \quad \mathcal{O}_W(1) \cong \pi_r^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \pi_s^*\mathcal{O}_{\mathbb{P}^s}(1).$$

In other words, for the irreducible divisor $E = \mathbb{P}^r \times \mathbb{P}^s \hookrightarrow X$, we have

$$\mathcal{O}_E(E) \cong \mathscr{N}_{E/X} \cong \mathcal{O}_W(-1) \cong \pi_r^*\mathcal{O}_{\mathbb{P}^r}(-1) \otimes \pi_s^*\mathcal{O}_{\mathbb{P}^s}(-1).$$

□

Lemma 3.6. *Denote by ξ and h the pull-backs of the classes $\xi, h \in \operatorname{Pic}(Y)$ to X. The vector space $\mathrm{N}^1(X)_{\mathbb{R}}$ has dimension 3, generated by $[E]$ and the nef classes ξ and h.*

Proof. Indeed, by Lemma 2.2 the morphism $\pi \colon X \to Y$ induces an isomorphism

$$\operatorname{Pic}(X) \cong \operatorname{Pic}(Y) \oplus \mathbb{Z} \cdot \mathcal{O}_X(E).$$

By Equation (4), this implies that

$$\operatorname{Pic}(X) = \mathbb{Z} \cdot \xi \oplus \mathbb{Z} \cdot h \oplus \mathbb{Z} \cdot [E].$$

□

Lemma 3.7. *We have the following multiplication table:*

$$h \cdot \ell = 1, \quad h \cdot \ell' = 0, \quad h \cdot \ell'' = 0, \tag{9}$$

$$\xi \cdot \ell = 0, \quad \xi \cdot \ell' = 1, \quad \xi \cdot \ell'' = 0, \tag{10}$$

$$[E] \cdot \ell = -1, \quad [E] \cdot \ell' = 1, \quad [E] \cdot \ell'' = -1. \tag{11}$$

Proof. Indeed, let $L \subset P$ be the line that gives $\ell \in \mathrm{NE}(Y)$ such that $\varepsilon_*(\ell) = \ell$. Recall that $j \colon P \to Y$ was the inclusion. Let $L' \subset \mathbb{P}^s$ be the line that corresponds to L under the isomorphism $P \cong \mathbb{P}^s$. Then

$$h \cdot \ell = \varepsilon^*(h) \cdot \ell = h \cdot \varepsilon_*(\ell) = \pi^*\mathcal{O}(1) \cdot j_*([L]) = \mathcal{O}(1) \cdot [L'] = 1.$$

$$h \cdot \ell' = h \cdot \varepsilon_*(\ell') = 0, \quad h \cdot \ell'' = h \cdot \varepsilon_*(\ell'') = 0.$$

Moreover,

$$\xi \cdot \ell = \xi \cdot \varepsilon_*(\ell) = 0, \quad \xi \cdot \ell' = \xi \cdot \varepsilon_*(\ell') = 1, \quad \xi \cdot \ell'' = \xi \cdot \varepsilon_*(\ell'') = 0,$$

and if $\phi \colon E = \mathbb{P}^r \times \mathbb{P}^s \hookrightarrow X$ is the inclusion, then (because $\pi_{r,*}(\ell) = 0$) we have

$$[E] \cdot \ell = \mathcal{O}_E(E) \cdot \phi^*(\ell) = (\pi_r^*(\mathcal{O}(-1)) + \pi_s^*(\mathcal{O}(-1))) \cdot [l] =$$
$$\mathcal{O}_{\mathbb{P}^r}(-1) \cdot \pi_{r,*}[l] + \mathcal{O}_{\mathbb{P}^s}(-1) \cdot \pi_{s,*}[l] = (\mathcal{O}_{\mathbb{P}^r}(-1) \cdot \pi_{r,*}[l]) - 1 = -1.$$

In a similar way, one proves that $[E] \cdot \ell' = 1$. Finally,

$$[E] \cdot \ell'' = \mathcal{O}_E(E) \cdot \phi^*(\ell'') = -1$$

because the projection $E \to P$ corresponds to the projection $\mathbb{P}^s \times \mathbb{P}^r \to \mathbb{P}^s$, and since this contracts the line ℓ'', $\ell'' = [L'']$ for some line L'' contained in a fiber $\{x\} \times \mathbb{P}^r \subset E \to \mathbb{P}^s$.

□

3.4 The cone of curves in X

The goal of this subsection is to prove the following:

Proposition 3.8. *We have*

$$\mathrm{NE}(X) = \overline{\mathrm{NE}}(X) = \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell' + \mathbb{R}^+ \cdot \ell''.$$

Proof. Let $C \subset X$ be an irreducible curve, and let

$$[C] = a\ell + a'\ell' + a''\ell''$$

be its class in $\mathrm{NE}(X)$.

- *The subvariety P is defined as the complete intersection of $r+1$ divisors with class $\xi - h$.*
- *This implies that if C is not contained in E, the class $\xi - h - [E]$ on X has nonnegative intersection with C.*

We conclude that

$$h \cdot C = a \geq 0, \quad a' = \xi \cdot C \geq 0, \quad a'' = (\xi - h - [E]) \cdot C \geq 0.$$

- *If $C \subset E$, then $a' = 0$. Moreover, a and a'' are nonnegative in this case.*

The conclusion is that $[C] \in \mathbb{R}^+ \cdot \ell + \mathbb{R}^+ \cdot \ell' + \mathbb{R}^+ \cdot \ell''$, which proves the proposition. □

Lemma 3.9. *We have $X = X_{r,s} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s}(1,1))$.*

4 Example III: · a flip

Let us prove the statements made in Remark 3.3. We begin with a

Lemma 4.1. *Let Y be a Fano variety and let $D_1, \dots, D_r$ be nef divisors on Y such that $-K_Y - D_1 \dots - D_r$ is ample. Then $X = \mathbb{P}(\bigoplus_{i=1}^r \mathcal{O}_Y(D_i))$ is a Fano variety.*

Proof. Indeed,

$$\det\left(\bigoplus_{i=1}^r \mathcal{O}_Y(D_i)\right) = \mathcal{O}_Y(\sum_i D_i),$$

thus by Lemma 1.2, we have $K_X = (r+1)\cdot\mathcal{O}_X(1) + \pi^*(-K_Y - D_1 \cdots - D_r) \in \mathrm{N}^1(X)$. □

Corollary 4.2. *Assume that $r < s$. The variety $Y_{r,s}$ is a Fano variety.*

Proof. It suffices to observe that $\mathscr{E} = \mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(r+1)}$ is a direct sum of nef divisors, and that

$$-K_{\mathbb{P}^s} - \mathcal{O}_{\mathbb{P}^s}(r+1) = \mathcal{O}_{\mathbb{P}^s}(s+1) - \mathcal{O}_{\mathbb{P}^s}(r+1) = \mathcal{O}_{\mathbb{P}^s}(s-r)$$

is ample because $s > r$. □

Proposition 4.3. *Let $R = \mathbb{R}^+ \cdot \ell$.*

1. *The contraction*
$$c_R \colon Y_{r,s} \to \widehat{Y}_{r,s}$$
of the $K_{Y_{r,s}}$-negative extremal ray R is the morphism associated with the base-point-free linear system $|\mathcal{O}_{Y_{r,s}}(1)|$.

2. *One has $L(R) = P \subset Y_{r,s}$.*

3. *The morphism c_R is a* small contraction.

Lemma 4.4. *Continue to assume that $r < s$ and consider the variety $Y_{s,r}$. Let $R = R^+ \cdot \ell \subset \mathrm{NE}(Y_{s,r})$. Then $K_{Y_{s,r}} \cdot R > 0$.*

Proof. By Lemma 1.2, we have $K_{Y_{s,r}} = -(r+1)\cdot\mathcal{O}_{Y_{s,r}}(1) + \pi^*(K_{\mathbb{P}^r} + \mathcal{O}_{\mathbb{P}^s}(s+1))$. Therefore,

$$K_{Y_{s,r}} \cdot \ell = \pi^*(K_{\mathbb{P}^r} + \mathcal{O}_{\mathbb{P}^s}(s+1)) \cdot \ell = \pi^*(\mathcal{O}_{\mathbb{P}^r}(s-r)) \cdot \ell = \mathcal{O}_{\mathbb{P}^s}(s-r) \cdot \pi_*(\ell) = s - r > 0.$$

□

In other words, where $Y_{r,s}$ was Fano and $\ell \in \mathrm{NE}(Y_{r,s})$ was $K_{Y_{r,s}}$-negative, the variety $Y_{s,r}$ is *not* a Fano variety, and the class $\ell \in \mathrm{NE}(Y_{r,s})$ is *$K_{Y_{s,r}}$-positive.*

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