# Alterations

Notes by Olivier de Gaay Fortman, January 24, 2023

## 1 Introduction

1.1 **General idea.** The goal of this notes is to explain the main ideas behind the proof of De Jong's alteration theorem [Jon96]. We also put the alteration theorem in some context, by providing applications and extensions.

**Notation 1.1.** A *variety* is an integral separated scheme of finite type over a field. A *modification* of a variety $X$ is a proper birational morphism $X' \to X$ with $X'$ a variety. An *alteration* of $X$ is a proper dominant map $X' \to X$ where $X'$ is a variety with $\dim(X') = \dim(X)$.

**Examples 1.2.** 1. Blow-ups are modifications.

2. Modifications are alterations.

3. Finite field extensions are alterations.

These notions are used in geometric situations, when one wants to simplify such a situation by 'modifying' or 'altering' the varieties under consideration. The idea is to get to this new situation while staying as close to the geometry of original variety as possible.

**Examples 1.3.** 1. (From NC to SNC.) Let $D$ be a normal crossings divisor on a noetherian scheme $S$. There exists a blow-up $S' \to S$ in an idea with support in $D$ such that $\varphi^{-1}(D)_{\text{red}}$ is a strict normal crossings divisor.

2. (Extending families of curves.) Let $Y$ be a projective variety, $U \subset Y$ a non-empty open subset, $n \in \mathbb{Z}_{\geq 3}$ and

$$(\mathscr{C}_0 \to U, \quad \sigma_1, \ldots, \sigma_n \colon U \to \mathscr{C}_0)$$

be a smooth $n$-pointed curve of genus $g \geq 0$.

*Claim:* There exists a generically étale alteration $\varphi\colon Y' \to Y$ such that for $U' = \varphi^{-1}(U) \subset Y'$, the base change

$$\mathscr{C}_0' \to U'$$

extends to a stable $n$-pointed curve $\mathscr{C}' \to Y'$. *Proof:* We get a morphism

$$U \to \mathscr{M}_{g,n}$$

to the stack of smooth stable $n$-pointed curves of genus $g$. Choose $\ell \geq 3$ a prime number unequal to $\mathrm{char}(k)$, and let

$$U' \subset U \times_{\mathscr{M}_{g,n}} M^{\ell}_{g,n}$$

be an irreducible component. Then $U'$ non-empty and finite étale over $U$. Put

$$Y' = \overline{\mathrm{Im}\left(U' \to Y \times \overline{M}^{\ell}_{g,n}\right)} \subset Y \times \overline{M}^{\ell}_{g,n}.$$

Then $Y' \to Y$ is a generically étale alteration by a projective variety, and the composition

$$Y' \to \overline{M}^{\ell}_{g,n} \to \overline{\mathscr{M}_{g,n}}[1/\ell] \to \overline{\mathscr{M}_{g,n}}$$

gives the desired family of curves. □

What about resolving singularities?

## 1.2 Resolution of singularities.

**Definition 1.4.** Let $(X, Z)$ be a pair where $X$ is a variety over a field and $Z \subset X$ a closed subscheme. The pair $(X, Z)$ admits an *altered desingularization* (or *altered resolution of singularities*) if there exists an alteration

$$\varphi \colon X' \to X,$$

a smooth projective variety $\overline{X}'$ over $k$, an open immersion $j \colon X' \to \overline{X}'$ and a strict normal crossings divisor $D \subset \overline{X}'$ such that $D = \overline{j(Z')} \cup \overline{X}' \setminus X' \subset \overline{X}'$. We say that $(X, Z)$ admits a *desingularization* if such an alteration exists with $\varphi$ birational.

Picture:

$$\begin{array}{ccccc}
\varphi^{-1}(Z) & & \longrightarrow & & D \\
\downarrow & \searrow & & & \downarrow \\
 & & X' & \longrightarrow & \overline{X}' \\
\downarrow & & \downarrow & & \\
Z & \longrightarrow & X. & &
\end{array}$$

We have seen, so far:

**Theorem 1.5** (Hironaka)**.** *Let $X$ be a variety over a field $k$ of characteristic zero, and let $Z \subset X$ be a proper closed subscheme. Then $(X, Z)$ admits a desingularization.*

In this talk, and the next, Morten and I will prove:

**Theorem** (Theorem (A). De Jong)**.** *Let $X$ be a variety over a field $k$, and let $Z \subset X$ be a proper closed subscheme. Then $(X, Z)$ admits an altered desingularization. If $k$ is perfect then the alteration $\varphi \colon X' \to X$ may be chosen generically étale.*

**Remarks 1.6.** 1. If $k$ is perfect, then $\overline{X}'$ is smooth over $k$. (But there are regular varieties which are not smooth: think of $Z(y^p - x^2 - t) \subset \mathbb{A}^2_{\mathbb{F}_p(t)}$.)

2. One obtains:

**Corollary 1.7.** *Any variety $X' \to X$ admits an alteration by a regular variety. If $X$ is projective then there exists an alteration $X' \to X$ with $X'$ projective and regular.*

3. The subscheme $Z \subset X$ is actually used the proof, which relies on an inductive argument. It is not clear how to prove Corollary 1.7 directly.

4. The proof shows that the structural morphism $\overline{X}' \to \operatorname{Spec} k$ factors over a finite field extension $k \subset k'$ such that $\overline{X}'$ is geometrically irreducible over $k'$.

## 2 Applications and extensions

2.1 **Applications.** Using Theorem (A), one can prove, for instance, that:

1. For any variety over a perfect field $k$, there exists:

(a) a simplicial scheme $X_\bullet$ projective and smooth over $k$;

(b) a strict normal crossings divisor $D_\bullet$ in $X_\bullet$; we put $U_\bullet$, and

(c) an augmentation $a\colon U_\bullet \to X$ which is a proper hypercovering of $X$.

2. We have [Del74]:

$$\mathrm{H}^i_{et}(X \otimes \bar{k}, \mathbb{Q}_\ell) \cong \mathbf{H}^i_{et}(U_\bullet \otimes \bar{k}, \mathbb{Q}_\ell).$$

If $k$ is finite, then the eigenvalues of Frobenius on $\mathrm{H}^i_{et}(X \otimes \bar{k}, \mathbb{Q}_\ell)$ occur as eigenvalues of Frobenius on some cohomology group of some smooth projective variety, and hence are Weil numbers (compare [Del80]).

3. If $k = \mathbb{C}$, then [Del74]:

$$\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \cong \mathbf{H}^i(U_\bullet(\mathbb{C}), \mathbb{Q}).$$

Moreover, Theorem (A) suffices to construct the mixed Hodge structure on $\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q})$.

4. Similarly, Theorem (A) can be used to define crystalline cohomology of $X$, when $X$ is defined over a perfect field of characteristic $p > 0$. One obtains a finite-dimensional cristalline cohomology (but it is not clear that the result is independent of the choice of $(X_\bullet, D_\bullet, a)$).

2.2 **Extensions.** One has the following extension of Theorem (A):

**Theorem 2.1** (Gabber). *Let $X$ be a variety over a field $k$, and $\ell \neq \mathrm{char}(k)$ a prime number. The alteration $X' \to X$ with $X$ regular can be chosen in such a way that $[K(X') \colon K(X)]$ is prime to $\ell$.*

This has useful consequences. For instance, if $X$ is a smooth projective variety over a perfect field $k$ of characteristic $p$, and $Z \subset X$ is a closed subvariety of codimension $k$, let $Z' \to Z$ be an alteration with $Z'$ smooth projective over $k$. Consider the composition $f \colon Z' \to Z \to X$. One may then define $[Z] := \frac{1}{d} \cdot f_*(1) \in \mathrm{H}^{2k}_{et}(X_{k^{sep}}, \mathbb{Z}_\ell(k))$. Here $d = [K(Z') \colon K(Z)]$ and

$$f_* \colon \mathbb{Z}_\ell = \mathrm{H}^0(Z_{k^{sep}}, \mathbb{Z}_\ell) \to \mathrm{H}^{2k}_{et}(X_{k^{sep}}, \mathbb{Z}_\ell(k))$$

is the push-forward on cohomology attached to the morphism of smooth varieties $f$ [Mil80, Section IV, §11]. Since any two alterations by smooth projective varieties can be covered by a third, this class does not depend on the alteration chosen.

The most general known statement seems to be the following. Given a scheme $X$, by *char*$(X)$ we denote the set of all primes $p$ with $p = \mathrm{char}(k(x))$ for some $x \in X$. Let $\mathcal{P}$ be a set of primes, possibly empty. By a $\mathcal{P}$-alteration we mean an alteration such that if $d = \prod_i p_i^{k_i}$ is the degree of the alteration, then $p_i \in \mathcal{P}$ for all $i$.

**Example 2.2.** Let $p$ be a prime number. Then a $\{p\}$-alteration of a variety over a field of characteristic $p$ is an alteration $X' \to X$ of degree $p^r$ for some $r \geq 0$.

**Theorem 2.3** (De Jong, Gabber, Illusie, Temkin). *Let $X$ and $S$ be Noetherian integral schemes, $f \colon X \to S$ a morphism of finite type. If any alteration of $S$ can be desingularized by a* $\mathrm{char}(S)$*-alteration, then both $X$ and $f$ can be desingularized by a* $\mathrm{char}(X)$*-alteration.*

**Corollary 2.4.** *Let $X$ be a variety over a field of characteristic $p \geq 0$. Let $\mathcal{P}$ be any set of prime numbers such that $p \in \mathcal{P}$ if $p > 0$. Then $X$ is $\mathcal{P}$-resolvable. If $p = 0$ then $X$ admits a resolution of singularities.*

**Remark 2.5.** If $k$ is perfect then $X$ is separably $\mathcal{P}$-resolvable.

Let us be more precise. Let $P$ be a set of primes, and $S$ a Noetherian integral scheme. We say that $S$ is *universally $P$-resolvable* if for any alteration $Y \to S$ and any strict closed subset $Z \subset Y$ there exists a $P$-alteration $f \colon Y' \to Y$ such that $Y'$ is regular and $Z' = f^{-1}(Z)$ is a strict normal crossings divisor. The precise version of Theorem 2.3 is the following.

**Theorem 2.6** (De Jong, Gabber, Illusie, Temkin). *Let $X$ and $S$ be Noetherian integral schemes, $f \colon X \to S$ dominant of finite type. Let $Z \subset X$ be a strict closed subset, and assume that $S$ is universally $P$-resolvable for a set of primes $P$ with* $\mathrm{char}(S) \subset P$*. Then*

*(i) $X$ is universally $P$-resolvable.*

*(ii) There exist projective $P$-alterations $a\colon S' \to S$ and $b\colon X' \to X$, with regular sources, a quasi-projective morphism $f'\colon X' \to S'$ compatible with $(f, a, b)$, and snc divisors $W' \subset S'$ and $Z' \subset X'$, such that $Z' = b^{-1}(Z) \cup (f')^{-1}(W')$ and the morphism $(X', Z') \to (S', W')$ is log-smooth.*

*(iii) If $S = \operatorname{Spec} k$ and $k$ is a perfect field, then one can achieve in addition to (ii) that $a$ is an isomorphism and $b$ is separable.*

*Proof.* See [Tem17]. □

For any practical application, one should start with a class of schemes $S$ that satisfy the desingularization property as above. Recall that a Noetherian ring is *quasi-excellent* if it has geometrically regular formal fibres and if any finite type algebra over it has closed singular set. A locally noetherian scheme is quasi-excellent if it admits an affine open covering by spectra of quasi-excellent rings. Remark that any algebraic scheme is excellent [Liu02, Chapter 8, Corollary 2.40].

Reduced, separated, Noetherian, quasi-excellent schemes of dimension $\leq 3$ admit a modification by a regular scheme [CP19]. Hence, to apply Theorem 2.6, we can take $S$ to be a three-dimensional scheme having these properties.

Finally, we recall the relation between quasi-excellent schemes and resolutions of singularities.

**Proposition 2.7** (Grothendieck)**.** *Let $X$ be a locally noetherian scheme. Suppose that, for every integral scheme $Y$ finite over $X$, one can resolve the singularities of $Y$. Then $X$ is a quasi-excellent scheme.*

*Proof.* See [DG61, §7]. □

Grothendieck asks whether the converse holds:

**Conjecture 2.8** (Grothendieck)**.** *Let $X$ be a reduced locally Noetherian scheme. If $X$ is quasi-excellent, then $X$ admits a desingularization (and hence the same holds for any reduced scheme $Y$ locally of finite type over $X$).*

## 3 Outline of the proof of Theorem (A)

As we explained above, the proof of Theorem (A) will be done in two talks. In this talk, I present the first half of the proof of the theorem, and explain what will be next.

3.1 **Outline of the proof.** The idea is as follows.

1. By applying suitable alterations, reduce Theorem (A) to the case where $k = \bar{k}$, $X$ is normal and projective, and $Z \subset X$ is the support of a Cartier divisor $D \subset X$.

2. Modify $X$ (i.e. define a projective modification of $X$) in order to construct a suitable family of curves $X \to Y$ over a projective variety $Y$, smooth over a non-empty open $U \subset Y$, together with $n \geq 3$ sections $\sigma_i\colon Y \to X$ such that $Z = \cup_i \sigma_i(Z)$.

3. Use the moduli stacks of curves $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, and the projectivity of the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$, to define an alteration $Y' \to Y$ and an extension of the smooth $n$-pointed curve $X_{U'} \to U'$ to a stable $n$-pointed curve $f\colon \mathscr{C} \to Y'$. Replace $Y$ by $Y'$.

4. Prove that the rational map $\beta\colon \mathscr{C} \dashrightarrow X$ extends to a morphism $\beta\colon \mathscr{C} \to X$, and that there is a closed subset $D \subset Y$ such that $\beta^{-1}(Z)$ is contained in the union of the images of the sections $\tau_i\colon Y \to \mathscr{C}$ and $f^{-1}(D)$, and such that $\mathscr{C} \to Y$ is smooth over $Y \setminus D$. Since $\beta$ is a modification, after replacing $X$ by $\mathscr{C}$ we may assume that $X \to Y$ is a stable $n$-pointed curve, smooth over $Y \setminus D$.

5. Apply the **induction hypothesis** to the pair $(Y, D)$: up to altering the pair $(Y, D)$, we may assume that $D \subset Y$ is a divisor with strict normal crossings, $f\colon X \to Y$ is a stable curve, smooth over $Y \setminus D$; and

$$Z = \bigcup_i \sigma_i(Y) \cup f^{-1}(D) \subset X,$$

where $\sigma_i, 1 \leq i \leq n$ are mutually disjoint sections into the smooth locus of $f$.

6. Modify $X$ to get that $\mathrm{Sing}(X)$ has codimension at least three in $X$.

7. Calculate what the singularities of $(X, Z)$ must look like: for a smooth point $x \in X$, $Z$ is a normal crossings divisor at $x$, and for $x \in \mathrm{Sing}(X)$, there are $2 \leq s \leq r \leq d-1$ such that

$$\mathcal{O}_{X,x}{}^\wedge = k[[u, v, t_1, \ldots, t_{d-1}]]/(uv - t_1 \cdot t_2 \cdot \cdots \cdot t_s),$$

$Z \subset X$ is defined by $t_1 \cdots t_r = 0$, and the irreducible components of $\mathrm{Sing}(X)$ are smooth.

8. For the blow up $X' \to X$ of $X$ in a component of $\mathrm{Sing}(X)$, show that the number of components of $\mathrm{Sing}(X')$ is one less than the number of components of $\mathrm{Sing}(X)$.

9. By repeatedly blowing up $(X, Z)$, we get to the situation that $X$ is smooth and $Z$ is a strict normal crossings divisor. This finishes the proof of Theorem (A).

## 4 Proof of Theorem (A): Part I

Let $X$ be a variety over a field $k$, with a proper closed subscheme $Z \subset X$.

**Lemma 4.1.** *To prove that an altered desingularization of $(X, Z)$ exists, we may replace $(X, Z)$ by $(X', Z')$ for any alteration $\varphi\colon X' \to X$, with $Z' = \varphi^{-1}(Z)$.*

Consider the following hypotheses:

**Conditions 4.2.** *1. The field $k$ is algebraically closed.*

*2. $X$ is projective and there exists a divisor $D \subset X$ such that $Z$ is the support of $D$.*

**Proposition** (R). *To prove Theorem (A), we may assume that Conditions 4.2.1-4.2.2 hold.*

*Proof.* 1. Let $\bar{k}$ be an algebraic closure of $k$, let $Y$ be an irreducible component of $X \times_k \bar{k}$, and let $C$ be the inverse image of $Z$ in $Y$. Suppose that we can find $f\colon Y' \to Y$ and $Y \hookrightarrow \overline{Y}$ as in the theorem. Then the altered desingularization

$$\left(C' \subset Y' \to Y \supset C, Y \hookrightarrow \overline{Y}\right) \tag{1}$$

is defined over a finite extension $k_1$ of $k$. In other words, we have

$$\left(C_1' \subset Y_1' \to Y_1 \supset C_1, Y_1 \hookrightarrow \overline{Y}_1\right)$$

over $k_1$ such that these give rise to (1) after base extension. The composition

$$X' = Y_1' \to Y_1 \hookrightarrow X \otimes k_1 \to X$$

defines the desired alteration $X' \to X$ of $X$.

2. Perform the following steps:

- Take a modification $\varphi\colon X_1 \to X$ with $X_1$ quasi-projective (Chow's lemma) and define $Z_1 = \varphi^{-1}(Z)$.
- Embed $j\colon X_1 \to \overline{X}_1$ into a projective variety $\overline{X}_1$ and define $\overline{Z}_1 = \overline{j(Z_1)} \cup \overline{X}_1 \setminus X_1$.
- Blow up $\overline{X}_1$ in the ideal sheaf of $\overline{Z}_1$.

If the resulting pair $(\overline{X}_1', \overline{Z}_1')$ admits an altered desingularization, the same holds for $(X, Z)$. □

**Remark 4.3.** By normalizing $X$, we may assume in addition that $X$ is a normal variety.

## 5 Proof of Theorem (A)

In the sequel, a *curve* over a field $k$ is a geometrically connected algebraic scheme $C$ over $k$ which is equidimensional of dimension one.

**Theorem** (Theorem (B)). *Let $X$ be a projective variety over an algebraically closed field $k$, and let $Z \subset X$ be the support of a divisor $D \subset X$. There exists an alteration $X' \to X$, a closed subscheme $Z' \subset X'$ which is the support of a divisor $D' \subset X'$, a morphism of projective varieties $f\colon X' \to Y'$ such that:*

1. *All fibres are curves.*
2. *The smooth locus of $f$ is dense in all fibres.*
3. *The generic fibre of $f$ is smooth.*

4. *The morphism $f|_{Z'}\colon Z' \to Y'$ is finite and generically étale.*

5. *For all geometric points $\overline{Y} \in Y'$, and any irreducible component $C'$ of $X'_{\overline{Y}} = f^{-1}(\overline{Y})$, we have*
$$\#\,|\mathrm{sm}(X'/Y') \cap C' \cap Z'| \geq 3.$$

6. *There are sections $\sigma_i\colon Y' \to X'$, $i = 1, \ldots, n$, such that $Z' = \cup_i \sigma_i(Y')$.*

*Sketch of the proof of Theorem (A), assuming Theorem (B).* By Proposition (R) and Theorem (B), we may assume that there exists a morphism of projective varieties
$$f\colon X \to Y$$
satisfying the assumptions in Theorem (B). Let
$$U = \{y \in Y \mid X_y \text{ is smooth over } y \text{ and } \sigma_i(y) \neq \sigma_j(y) \text{ for } i \neq j\}.$$
Note that $U \neq \emptyset$ and that $n \geq 3$. Let $g$ be the genus of $X_y$ for $y \in U$. Let
$$Y' \to Y$$
be a generically étale alteration by a projective variety $Y'$ such that $U' \to \mathscr{M}_{g,n}$ extends to $Y' \to \overline{\mathscr{M}}_{g,n}$ (see Example 1.3).

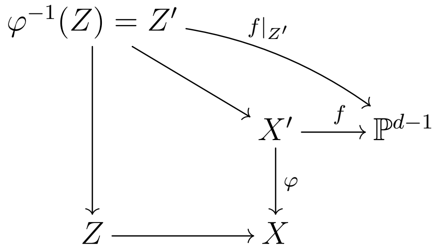
Replace $Y$ by $Y'$ and $X$ by the closed subscheme of $X \times_Y Y'$ given by dividing the $\mathcal{O}_{Y'}$-torsion out of $\mathcal{O}_{X \times_Y Y'}$. Therefore, we may assume that there exists a stable $n$-pointed curve
$$(\mathscr{C}, \tau_1, \ldots, \tau_n) \to Y,$$
a non-empty open subscheme $U \subset Y$ and an isomorphism
$$\beta\colon \mathscr{C}_U \xrightarrow{\sim} X_U$$
mapping each section $\tau_i|_U$ to the section $\sigma_i|_U$.

**Theorem** (C)**.** *There are modifications $Y' \to Y, X' \to X$ such that the rational map $\beta\colon \mathscr{C} \dashrightarrow X$ extends to a morphism $\beta\colon \mathscr{C}' \to X'$ and such that there is a closed subset $D' \subset Y'$ such that*
$$\beta^{-1}(Z') \subset \tau_1(Y') \cup \cdots \cup \tau_n(Y') \cup g^{-1}(D')$$
*and such that $\mathscr{C}' \to Y'$ is smooth over $Y' \setminus D'$.*

Replace $X$ by $\mathscr{C}$ and $Z$ by $\tau_1(Y) \cup \cdots \cup \tau_n(Y) \cup f^{-1}(D)$. Then apply the induction hypothesis to $(Y, D)$. Modify $X$ to get $\mathrm{codim}(\mathrm{Sing}(X), X) \geq 3$. The completions of the singularities of $X$ look like
$$\{uv - t_1 \cdots t_s = 0\} \subset \mathbb{A}_k^{d+1},$$
$Z$ is defined by $t_1 \cdots t_r = 0$, and the irreducible components of $\mathrm{Sing}(X)$ are smooth. Blow these components up repeatedly to get that $X$ is smooth. Then modify $X$ by blow-ups in smooth centers to make $Z \subset X$ a strict normal crossings divisor (and note that $X$ remains smooth [Har77, Theorem II.8.24]). □

## 6 Proof of Theorem (B)

**Lemma** (Key Lemma I)**.** *Let $X$ be a normal projective variety over an algebraically closed field $k$, and let $Z \subset X$ be the support of a divisor $D \subset X$. Then there exists a diagram*



*where $\varphi$ is the blow-up of $X$ in a finite set of closed points $S \subset \mathrm{Reg}(X)$ with $S \cap Z = \emptyset$, having the following properties:*

1. *All fibres of $f$ are non-empty, equidimensional schemes of dimension one.*

2. *The smooth locus of $f$ is dense in all fibres of $f$.*

3. *The morphism $f|_{Z'} \colon Z' \to \mathbb{P}^{d-1}$ is finite and étale over an open subscheme of $\mathbb{P}^{d-1}$.*

4. *At least one fibre of $f$ is smooth.*

**Lemma** (Key Lemma II)**.** *Let $f \colon X \to Y$ be a morphism of smooth projective varieties over an algebraically closed field $k$ such that all fibres are* curves*. Suppose that the smooth locus of $f$ is dense in all fibres. There exists a divisor $H \subset X$ such that*

*($\alpha$) $f|_H \colon H \to Y$ is finite and generically étale, and*

*($\beta$) for all geometric points $\overline{Y} \in Y$, and any irreducible component $C$ of $X_{\overline{Y}} = f^{-1}(\overline{Y})$, we have*

$$\# |\mathrm{sm}(X/Y) \cap C \cap H| \geq 3.$$

*Proof of Theorem (B) assuming Key Lemma's I and II.* Let $(X, Z)$ be as in the statement of the theorem. Replace $X$ by a modification to get into the situation of Key Lemma I. Thus, we get a family of curves

$$f \colon X \to Y$$

satisfying $1-4$. By Key Lemma II, there exists a divisor $H \subset X$ satisfying $(\alpha)-(\beta)$. It suffices to prove the theorem for the pair $(X, Z \cup H)$. Thus we may replace $Z$ by $Z \cup H$. We know have that for all geometric points $\overline{Y} \in Y$, and any irreducible component $C$ of $X_{\overline{Y}}$, have

$$\# |\mathrm{sm}(X/Y) \cap C \cap Z| \geq 3. \qquad (2)$$

Let $Z = \bigcup_i Z_i$ be the decomposition into irreducible components. Choose a finite Galois extension $k(Y) \subset L$ such that $k(Z_i)$ may be embedded over $k(Y)$ into $L$ for all $i$ (this is possible by 3).

Let $Y'$ be the normalization of $Y$ in the field $L$. Then $Y' \to Y$ is a finite generically étale alteration of $Y$. Let $(X', Z')$ be the strict transform of $(X, Z)$. We see that

$$Z' = Z'_1 \cup \cdots \cup Z'_n,$$

with $Z'_i \to Y'$ finite and birational, and where $n \geq 3$ by (2). Hence $Z'_i \to Y$ is an isomorphism ($Y$ is normal). In this way, we obtain sections

$$\sigma_i \colon Y \to X, \quad i = 1, \ldots, n$$

of $f$ such that $Z = \cup_{i=1}^{n} \sigma_i(Y)$, with $n \geq 3$. This proves Theorem (B). □

## References


[CP19] Vincent Cossart and Olivier Piltant. "Resolution of singularities of arithmetical threefolds". In: *Journal of Algebra* 529 (2019), pp. 268–535.

[Del74] Pierre Deligne. "Théorie de Hodge. III". In: *Publications Mathématiques de l'IHÉS* 44 (1974), pp. 5–77.

[Del80] Pierre Deligne. "La conjecture de Weil. II". In: *Publications Mathématiques de l'IHÉS* 52 (1980), pp. 137–252.

[DG61] Jean Dieudonné and Alexander Grothendieck. "Éléments de Géométrie Algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie". In: *Publications Mathématiques de l'IHÉS* (1961–1967).

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics 52. Springer, 1977, pp. xvi+496.

[Jon96] Aise Johan de Jong. "Smoothness, semi-stability and alterations". In: *Publications Mathématiques de l'IHÉS* 83 (1996), pp. 51–93.

[Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Vol. 6. Oxford Graduate Texts in Mathematics. Oxford University Press, 2002, pp. xvi+576.

[Mil80] James Milne. *Étale Cohomology*. Princeton Mathematical Series 33. Princeton University Press, 1980, pp. xiii+323.

[Tem17] Michael Temkin. "Tame distillation and desingularization by $p$-alterations". In: *Annals of Mathematics* 186.1 (2017), pp. 97–126.