Leibniz University Hannover — October – February 2024
The geometry and arithmetic of cubic hypersurfaces
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Exam of March 6, 2024

Notation. Throughout the exam, k will denote a field.

Exercise 1. Let n, d be positive integers. Let $X \subset \mathbb{P}^{n+1} := \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree d and dimension n, with canonical bundle $\omega_X = \Omega_X^n$.
1.1 (8 pts.) For which values of (d, n) is ω_X ample, anti-ample, resp. trivial? Prove this.
1.2 (8 pts.) Let $r \in \mathbb{Z}$ with $r < d$. Prove that the natural map

$$H^q(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(r)) \longrightarrow H^q(X, \mathcal{O}_X(r))$$

is bijective for $q < n$, and injective for $q \leq n$.
1.3 (8 pts.) Let $d = 3, n \geq 3$. Prove that the natural map $H^1(\mathbb{P}^{n+1}, \Omega^1_{\mathbb{P}^{n+1}}) \to H^1(X, \Omega^1_X)$ is an isomorphism.
1.4 (8 pts.) Assume $k = \mathbb{C}$. Provide a natural map

$$c_1 \colon \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}),$$

and prove that c_1 is an isomorphism whenever $n \geq 3$.

Exercise 2. (24 pts.) Write down the Hodge diamond of a smooth cubic threefold $X \subset \mathbb{P}^4_{\mathbb{C}}$, and justify your answer.

Exercise 3. **3.1** (8 pts.) Give the definition of a locally noetherian formal scheme.
3.2 (8 pts.) Let A and B be adic rings. Show that

$$\mathrm{Hom}(\mathrm{Spf}(B), \mathrm{Spf}(A)) = \mathrm{Hom}_{\mathrm{cont}}(A, B).$$

3.3 (8 pts.) Give the statement of Grothendieck's existence theorem and sketch its proof.
3.4 (8 pts.) Let A be an adic noetherian ring, I an ideal of definition, $Y = \mathrm{Spec}(A)$, $\widehat{Y} = \mathrm{Spf}(A)$. Provide schemes X, Z of finite type over Y, such that the natural map

$$\mathrm{Hom}_Y(X, Z) \longrightarrow \mathrm{Hom}_{\widehat{Y}}(\widehat{X}, \widehat{Z})$$

is not an isomorphism.

Exercise 4. In this exercise, we let $k = \mathbb{C}$.
4.1 (8 pts.) Provide the statement of the infinitesimal Torelli theorem for hypersurfaces.
4.2 (8 pts.) Let $\mathcal{X} \to U$ be the universal family of smooth cubic hypersurfaces of dimension $n \geq 2$. For $0 \in U$, let $X = \mathcal{X}_0$.

Define the Kodaira–Spencer map $\rho \colon T_{U,0} \to H^1(X, T_X)$, and show that ρ is surjective.

Exercise 5. All varieties in this exercise are defined over k.
5.1 (8 pts.) Let $X = V(F) \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d. Assume that, for some $i \in \{0, \dots, n+1\}$, the degree of F as a polynomial in x_i is less than or equal to $d-2$. Show that X is singular.
5.2 (8 pts.) Assume that the characteristic of k is zero. Let $S \subset \mathbb{P}^3$ be a smooth hypersurface of degree $d \geq 1$ and dimension two. Show that $\mathrm{Pic}(S)$ is torsion-free.
5.3 (8 pts.) Sketch a proof of the fact that if a smooth cubic hypersurface X of even dimension $n = 2m$ contains two complementary linear subspaces of dimension m, then X is rational.

Exercise 6. Let X be a scheme, smooth over k. Let $D = k[t]/(t^2)$. Let $\mathrm{Def}_D(X)$ be the space of infinitesimal deformations of X.
6.1 (8 pts.) Assuming Grothendieck's theorem on formal smoothness, prove that if X is affine, then $\mathrm{Def}_D(X) = 0$.
6.2 (8 pts.) Which coherent sheaf cohomology group parametrizes $\mathrm{Def}_D(X)$?
6.3 (8 pts.) Let E be an elliptic curve over k. Calculate the dimension of $\mathrm{Def}_D(E)$.
6.4 (8 pts.) Let Y be a closed subscheme of X. If Y is affine, prove that every abstract deformation of Y over the dual numbers can be realized as an embedded deformation.
6.5 (8 pts.) Let C is a smooth projective plane curve of degree $d \geq 5$. Show that there are abstract deformations of C over the dual numbers that cannot be realized as embedded deformations in $\mathbb{P}^2$.
6.6 (8 pts.) Prove that any infinitesimal deformation of a smooth cubic hypersurface as a variety over k is again a smooth cubic hypersurface.

Exercise 7. Recall that a k-algebra A is étale over k if $A \otimes_k \bar{k}$ is isomorphic as a $\bar{k}$-algebra to a finite product of copies of $\bar{k}$. A ring map $A \to B$ is étale if it is flat, of finite type, and for each prime ideal $\mathfrak{p} \subset A$, the algebra $B \otimes_A k(\mathfrak{p})$ is étale over $k(\mathfrak{p})$.
7.1 (8 pts.) Let $A \to B$ be a standard étale morphism of rings. Prove that $A \to B$ is étale.
7.2 (8 pts.) Let X be a scheme, and let F be a sheaf on the étale site of X. Give the definition of the étale cohomology groups $H^i(X_{\text{ét}}, F)$.
7.3 (8 pts.) Let X be a smooth projective connected curve over an algebraically closed field k. Let $n \in \mathbb{Z}_{\geq 2}$ be coprime to the characteristic of k. Give a natural isomorphism

$$H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathrm{Pic}^0(X)[n], \tag{1}$$

where $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ is the subgroup of isomorphism classes of line bundles on X which are of degree zero. To which abstract group are the groups in (1) isomorphic?

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