Algebraic Geometry II : Part 2

Lecture notes

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Contents

Chapter 1

Quasi-coherent sheaves on projective schemes

1.1 Lecture 14 : Quasi-coherent sheaves and projective spectra

Definition 1.1.1. A *graded ring* is a ring S with a decomposition

$$S = \oplus_{d \in \mathbb{Z}_{\geq 0}} S_d$$

of the underlying abelian group into abelian subgroups $S_d \subset S$, such that $S_d \cdot S_e \subset S_{d+e}$. A *$\mathbb{Z}$-graded ring* is a ring S with a decomposition $S = \oplus_{d \in \mathbb{Z}} S_d$ of the underlying abelian group into abelian subgroups $S_d \subset S$, such that $S_d \cdot S_e \subset S_{d+e}$.

Goal of this lecture: For a graded ring S, consider the scheme $X = \mathrm{Proj}(S)$, and define a functor

$$M \mapsto \widetilde{M}$$

from the category of graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules, as in the affine case.

Recall. *A* graded abelian group *is an abelian group M together with a decomposition $M = \oplus_{d \in \mathbb{Z}} M_d$ into abelian subgroups $M_d \subset M$.*

Recall. *Let $S = \oplus S_d$ be a graded ring, which is either graded or $\mathbb{Z}$-graded.*

(1) A graded S-module *is an S-module M with the structure of a graded abelian group $M = \oplus M_d$, such that the gradings of S and M are compatible in the sense that $S_d \cdot M_e \subset M_{d+e}$ for all $d, e \in \mathbb{Z}$.*

(2) An element $x \in M$ is called homogeneous *if $x \in M_d$ for some $d \in \mathbb{Z}$.*

(3) A graded submodule *of a graded S-module M is a submodule $N \subset M$ which is generated by homogeneous elements.*

(4) A morphism *of graded S-modules $\varphi \colon M \to N$ is a morphism of S-modules such that $\varphi(M_d) \subset N_d$ for $d \in \mathbb{Z}$.*

Question 1.1.2. *In which ways can you turn $R = \mathbb{Z}$ into a graded ring?*

Definition 1.1.3. Let $M = \oplus M_d$ be a graded S-module. For $n \in \mathbb{Z}$, define a new graded S-module $M(n)$ as follows:

$$M(n)_d := M_{d+n}, \qquad M(n) := \oplus M(n)_d.$$

In particular, we have the graded S-module $S(n)$ for $n \in \mathbb{Z}$.

Lemma 1.1.4. *Let S be a graded ring and M a graded S-module.*

(1) An S-submodule $N \subset M$ is a graded submodule if and only if $N = \oplus N_d$ for $N_d := N \cap M_d$.

(2) If $N \subset M$ is a graded submodule, then M/N is naturally a graded S-module.

(3) Let $\varphi \colon M \to N$ be a morphism of graded S-modules. Then the kernel, image and cokernel of φ are graded S-modules in a natural way.

Proof. (1) Consider a submodule $N \subset M$, and define $N_d = N \cap M_d$ for $d \in \mathbb{Z}$. By definition, N is graded if and only if N is generated by the submodules $N_d \subset N$ for $d \in \mathbb{Z}$. As $N_d \cap N_{d'} = 0$ for $d \neq d'$, this happens if and only if $N = \oplus N_d$.

(2) Define $(M/N)_d = \mathrm{Im}(M_d \to M/N)$. Then the natural map

$$\oplus (M/N)_d \longrightarrow M/N$$

is surjective. We need to show it is injective. In other words, we need to show, for $d \neq e \in \mathbb{Z}$, that $(M/N)_d \cap (M/N)_e = 0$. Let

$$x \in (M/N)_d \cap (M/N)_e.$$

There exists $m_d \in M_d$ and $m_e \in M_e$ which both have image $x \in M/N$. Hence,

$$m_d \equiv m_e \mod N.$$

In other words, $m_d - m_e \in N$. Since N is graded, we can write $m_d - m_e = \sum_{k \in \mathbb{Z}} n_k$ as a sum of homogeneous elements $n_k \in N_k$. We have $N_k \subset M_k$, and it follows that $n_k = 0$ for $k \neq d, e$, and that $m_d = n_d$ and $m_e = -n_e$. In particular, $m_d, m_e \in N$, so that $x = 0 \in M/N$.

(3) In view of item (2), it suffices to prove the statement for the kernel $\mathrm{Ker}(\varphi)$ of $\varphi \colon M \to N$. Indeed, we have $\mathrm{Im}(\varphi) = M/\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi) = N/\mathrm{Im}(\varphi)$. Thus, let us show that $K := \mathrm{Ker}(\varphi)$ is a graded S-module. Let $x \in K$. Write $x = \sum m_d$ for $m_d \in M_d$. Then

$$0 = \varphi(x) = \sum \varphi(m_d).$$

As $\varphi(m_d) \in N_d$, this implies $\varphi(m_d) = 0$ for each $d \in \mathbb{Z}$. Hence $m_d \in K$.

This proves the lemma. ∎

Remark 1.1.5. Let S be a graded ring, and M a graded S-module. Let $\mathfrak{p} \in \operatorname{Proj}(S)$. Consider the multiplicatively closed subset $T \subset S$ containing all homogeneous elements in $S \setminus \mathfrak{p}$. Then $T^{-1}M$ is naturally a graded $T^{-1}S$-module: we have

$$T^{-1}M = \oplus \left(T^{-1}M\right)_k, \qquad \text{with}$$
$$\left(T^{-1}M\right)_k = \left\{\frac{m}{t} \in T^{-1}M \colon m \text{ homogeneous of degree } k + \deg(t)\right\}.$$

Definition 1.1.6. Consider the notation in Remark 1.1.5. We define

$$M_{(\mathfrak{p})} \coloneqq \left(T^{-1}M\right)_0.$$

Notice that $M_{(\mathfrak{p})}$ is an $R_{(\mathfrak{p})}$-module in a natural way.

Definition 1.1.7. Let M be a graded S-module. Let $U \subset \operatorname{Proj}(S)$ be open, and define

$$\widetilde{M}(U) = \left\{(s(\mathfrak{p})) \in \prod_{\mathfrak{p} \in U} M_{(\mathfrak{p})} \colon \text{ condition } (\star) \text{ holds}\right\},$$

where $(\star)$ is the condition that for each $\mathfrak{p} \in U$, there exists an open neighbourhood $\mathfrak{p} \in V_{\mathfrak{p}} \subset U$ of $\mathfrak{p}$ in U, together with homogeneous elements $m \in M, f \in S$ of the same degree, such that for all $\mathfrak{q} \in V_{\mathfrak{p}}$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{f} \in M_{(\mathfrak{q})}$.

Proposition 1.1.8. *Let $X = \operatorname{Proj}(S)$ for a graded ring S, and let M be a graded S-module. The following assertions are true.*

(1) For all $\mathfrak{p} \in \operatorname{Proj}(S)$, we have a canonical isomorphism

$$\left(\widetilde{M}\right)_{\mathfrak{p}} \cong M_{(\mathfrak{p})}.$$

(2) Let $f \in S_+$ homogeneous, and consider the canonical isomorphism

$$\varphi \colon D_+(f) \xrightarrow{\sim} \operatorname{Spec} S_{(f)}.$$

Then there is a canonical isomorphism

$$\widetilde{M}|_{D_+(f)} \cong \varphi^* \left(\widetilde{M_{(f)}}\right).$$

Here, $M_{(f)}$ denotes the degree zero part of M_f (note that $M_{(f)}$ is an $S_{(f)}$-module in a natural way) and $\widetilde{M_{(f)}}$ is the affine tilde construction.

(3) The sheaf $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is noetherian and M finitely generated, then $\widetilde{M}$ is coherent.

Proof. (1). We have

$$\left(\widetilde{M}\right)_{\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in U\subset X} \widetilde{M}(U).$$

For $U \subset X$ open with $\mathfrak{p} \in U$, define a map

$$f_U\colon \widetilde{M}(U) \to M_{(\mathfrak{p})}, \quad (s(\mathfrak{q})) \mapsto s(\mathfrak{p}).$$

These maps are compatible with restrictions $\widetilde{M}(U) \to \widetilde{M}(V)$ for $\mathfrak{p} \in V \subset U$ open, and hence we get a well-defined map

$$f\colon \left(\widetilde{M}\right)_{\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in U\subset X} \widetilde{M}(U) \to M_{(\mathfrak{p})}. \tag{1.1}$$

We claim that (1.1) is an isomorphism. As for the surjectivity, let $m/f \in M_{(\mathfrak{p})}$ with m, f homogeneous, $f \notin \mathfrak{p}$ and $\deg(m) = \deg(f)$. Then for each $\mathfrak{q} \in D_+(f)$, put $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$. Then we get a section

$$s := (s(\mathfrak{q})) \in \widetilde{M}(D_+(f)),$$

and we have $f_{D_+(f)}(s) = m/f \in M_{(\mathfrak{p})}$. Thus, the map (1.1) is surjective.

To prove the injectivity, let $s, t \in (\widetilde{M})_{\mathfrak{p}}$ such that $f(s) = f(t)$. We can find an open neighbourhood $\mathfrak{p} \in U \subset X$ and $\bar{s}, \bar{t} \in \widetilde{M}(U)$ that map to $s, t \in (\widetilde{M})_{\mathfrak{p}}$. We have $\bar{s}(\mathfrak{p}) = \bar{t}(\mathfrak{p})$, and hence there exists an open neighbourhood $\mathfrak{p} \in V_{\mathfrak{p}} \subset U$ such that $\bar{s}|_{V_{\mathfrak{p}}} = \bar{t}|_{V_{\mathfrak{p}}}$. In particular, $s = t$, and we are done.

(2). Exercise.

(3). By (2), quasi-coherence is clear. If S is noetherian and M finitely generated, then $S_{(f)}$ is noetherian and $M_{(f)}$ is finitely generated, hence M is coherent by (2). □

Recall that for a scheme X and a sheaf $\mathcal{F}$ on X, one defines the support of $\mathcal{F}$ as

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Lemma 1.1.9. *For a graded S-module M,* $\mathrm{Supp}(\widetilde{M}) = \left\{\mathfrak{p} \in \mathrm{Proj}(S) \mid M_{(\mathfrak{p})} \neq 0\right\}$.

Proof. Clear from item (1) in Proposition 1.1.8. □

Lemma 1.1.10. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of graded S-modules. Then for each $d \in \mathbb{Z}$, the induced sequence*

$$0 \to A_d \to B_d \to C_d \to 0$$

is exact.

Proof. Everything apart from possibly the surjectivity of $B_d \to C_d$ is trivial. To prove this, let $x \in C_d$ and lift x to an element $y \in B$. Write $y = \sum_n y_n$. Then as y maps to x, y_n maps to zero for each $n \neq d$. Therefore, y_d maps to x, and $y_d \in B_d$. □

Lemma 1.1.11. *For a graded ring S, and $X = \operatorname{Proj}(S)$, the tilde construction $M \mapsto \widetilde{M}$ defines an exact functor from the category of graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of graded S-modules. Let $\mathfrak{p} \in \operatorname{Proj}(S)$. Then the sequence

$$0 \to (M_1)_{\mathfrak{p}} \to (M_2)_{\mathfrak{p}} \to (M_3)_{\mathfrak{p}} \to 0$$

is exact. In particular, in view of Lemma 1.1.10, the sequence

$$0 \to (M_1)_{(\mathfrak{p})} \to (M_2)_{(\mathfrak{p})} \to (M_3)_{(\mathfrak{p})} \to 0$$

is exact. By Proposition 1.1.8, we are done. □

Recall that, for a ring R and an R-module M, we have

$$\operatorname{Supp}(M) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}.$$

Lemma 1.1.12. *Let S be a graded ring and M, N graded S-modules.*

(1) Suppose that $\operatorname{Supp}(M) \subset V(S_+) \subset \operatorname{Spec} S$. Then $\widetilde{M} = 0$.

(2) Assume that $N_{>d} \cong M_{>d}$ for some $d \in \mathbb{Z}_{\geq 0}$. Then $\widetilde{M} \cong \widetilde{N}$.

Proof. (1). The assumption implies that $\operatorname{Supp}(M) \cap \operatorname{Proj}(S) = \emptyset$. Hence $M_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Proj}(S)$. In particular, $M_{(\mathfrak{p})} = 0$ for each $\mathfrak{p} \in \operatorname{Proj}(S)$. It follows that $(\widetilde{M})_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Proj}(S)$, see Proposition 1.1.8. Thus $\widetilde{M} = 0$.

(2). Since $M_{>d} \subset M$ is a graded submodule, the quotient $L \coloneqq M/M_{>d}$ is graded (see Lemma 1.1.4). Note that $\operatorname{Supp}(L) \subset V(S_+)$. Therefore, $\widetilde{L} = 0$ by item (1). From Lemma 1.1.11, it follows that the sequence

$$0 \to \widetilde{M_{>d}} \to \widetilde{M} \to \widetilde{L} \to 0$$

is exact. Hence $\widetilde{M_{>d}} \cong \widetilde{M}$. Consequently,

$$\widetilde{M} \cong \widetilde{M_{>d}} \cong \widetilde{N_{>d}} \cong \widetilde{N}.$$

We are done. □

Example 1.1.13. Let $X = \operatorname{Proj}(S)$ with $S = k[x_0, x_1]$, where k is a field. Let M be the graded S-module $M = k[x_0, x_1]/(x_0^2, x_1^2)$. Then $\widetilde{M} = 0$. Indeed, we have $S_+ = (x_0, x_1)$. If $M_{\mathfrak{p}} \neq 0$ for some $\mathfrak{p} \in \operatorname{Spec} S$, then $r \cdot 1 \neq 0$ for each $r \notin \mathfrak{p}$. Thus, $r \notin (x_0^2, x_1^2)$ for each $r \notin \mathfrak{p}$. Thus, $(x_0^2, x_1^2) \subset \mathfrak{p}$. Hence $(x_0, x_1) \subset \mathfrak{p}$, so that $\mathfrak{p} \in V(S_+)$.

1.1.1 Serre's twisting sheaf

Definition 1.1.14. Let S be a graded ring and $X = \mathrm{Proj}(S)$. For $n \in \mathbb{Z}$, define

$$\mathcal{O}_X(n) := \widetilde{S(n)}.$$

We call $\mathcal{O}_X(n)$ the n-th *twisting sheaf* (of Serre). If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, we put

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n),$$

and call $\mathcal{F}(n)$ the n-th *twist* of $\mathcal{F}$.

Proposition 1.1.15. *Let S be a graded ring such that S is generated by S_1 as an S_0-algebra. Let $X = \mathrm{Proj}(S)$. Then:*

(1) The sheaf $\mathcal{O}_X(n)$ is invertible for all $n \in \mathbb{Z}$.

(2) Let M, N be graded S-modules. There is a canonical isomorphism

$$\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}. \tag{1.2}$$

(3) For all graded S-modules M and $n \in \mathbb{Z}$, we have a canonical isomorphism

$$\widetilde{M}(n) \xrightarrow{\sim} \widetilde{M(n)}.$$

(4) We have canonical isomorphisms $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$ for $n, m \in \mathbb{Z}$.

Proof. (1). With respect to the identification $D_+(f) = \mathrm{Spec}\ S_{(f)}$, we have a canonical isomorphism

$$\mathcal{O}_X(n)|_{D_+(f)} \cong \widetilde{S(n)_{(f)}}$$

of sheaves on $\mathrm{Spec}\ S_{(f)}$. For $n \in \mathbb{Z}$ and $f \in S_1$, we have an isomorphism

$$S_{(f)} \longrightarrow S(n)_{(f)}, \qquad s \mapsto f^n \cdot s.$$

Thus, $\mathcal{O}_X(n)|_{D_+(f)}$ is a free $\mathcal{O}_X|_{D_+(f)}$-module of rank one. Since S is generated by S_1 over S_0, we have $S = \langle f \mid f \in S_1 \rangle$, hence $\mathrm{Proj}(S) = \cup_{f \in S_1} D_+(f)$.

(2). Indeed, let $f \in S_1$, and consider the canonical isomorphism $D_+(f) = \mathrm{Spec}\ S_{(f)}$. Using Proposition 1.1.8, we can define isomorphisms

$$\begin{gathered}\widetilde{M \otimes_S N}|_{D_+(f)} \cong (M \otimes_S N)_{(f)} \to M_{(f)} \otimes_{S_{(f)}} N_{(f)} \cong \widetilde{M} \otimes \widetilde{N}|_{D_+(f)}, \\ \frac{m \otimes n}{f^{\deg(m)+\deg(n)}} \mapsto \frac{m}{f^{\deg(m)}} \otimes \frac{n}{f^{\deg(n)}}.\end{gathered}$$

These isomorphisms agree on overlaps $D_+(f) \cap D_+(f)$, hence glue to give (1.2).

(3). This follows from (2), by taking $N = \mathcal{O}_X(n)$.

(4). This follows from (2), by observing that there are canonical isomorphisms

$$S(n) \otimes_S S(m) \xrightarrow{\sim} S(n+m), \quad s \otimes t \mapsto s \cdot t.$$

$\square$

1.2 Lecture 15 : Projective schemes

1.2.1 The associated graded module

In the affine case, we can recover M from $\mathcal{F} = \widetilde{M}$ by taking global sections. In the projective setting, this will not work, as for instance $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = k$. Instead, we will have to look at the various Serre twists $\mathcal{F}(d)$, $d \in \mathbb{Z}$.

Definition 1.2.1. Let S be a graded ring. Let $X = \operatorname{Proj}(S)$, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the *graded S-module associated to $\mathcal{F}$* as

$$\Gamma_*(\mathcal{F}) \coloneqq \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{F}(d)).$$

In particular, from X we get an associated $\mathbb{Z}$-graded ring

$$\Gamma_*(\mathcal{O}_X) \coloneqq \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)).$$

Question 1.2.2. *Note $R = \Gamma_*(\mathcal{O}_X)$ has a grading $R = \oplus_{d\in\mathbb{Z}} R_d$ indexed by the full set of integers $\mathbb{Z}$. Hence R is a $\mathbb{Z}$-graded ring in the sense of Definition 1.1.1. Is it always true that $R_d = 0$ for $d < 0$? In other words, is R actually a graded ring, or not?*

The S-module structures are defined as follows. Let M be a graded S-module. There is a canonical morphism

$$\alpha\colon M \longrightarrow \Gamma_*(\widetilde{M}). \tag{1.3}$$

To define α, let $m \in M_d$ for $d \in \mathbb{Z}$. We need to provide a global section $\alpha(m) \in \Gamma(X, \widetilde{M}(d))$. It suffices to provide sections $\alpha(m) \in \Gamma(D_+(f), \widetilde{M}(d))$ that agree on overlaps. We have

$$\Gamma(D_+(f), \widetilde{M}(d)) = (M(d))_{(f)},$$

and put

$$\alpha(m) \coloneqq \frac{m}{1} \in (M(d))_{(f)} = \left(M_{(f)}\right)_d.$$

This defines the map (1.3).

In particular, we get a canonical morphism

$$\beta\colon S \longrightarrow \Gamma_*(\widetilde{S}) = \Gamma_*(\mathcal{O}_X) = \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)). \tag{1.4}$$

This turns $\Gamma_*(\mathcal{O}_X)$ into a $\mathbb{Z}$-graded S-algebra (with compatible gradings). Moreover, for each $\mathcal{O}_X$-module $\mathcal{F}$, we have that $\Gamma_*(\mathcal{F})$ is a graded $\Gamma_*(\mathcal{O}_X)$-module in a canonical way. Indeed, by item (4) of Proposition 1.1.15, we have canonical isomorphisms

$$\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(e) = \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(e) \cong \mathcal{F}(d+e).$$

In particular, for $s \in \mathcal{O}_X(d)$ and $t \in \mathcal{F}(e)$, we get a canonical section $s \cdot t \in \mathcal{F}(d+e)$, which defines the graded $\Gamma_*(\mathcal{O}_X)$-module structure on $\Gamma_*(\mathcal{F})$. Via (1.4), we obtain the graded S-module structure on $\Gamma_*(\mathcal{F})$.

Proposition 1.2.3. *Let A be a ring, and $S = A[x_0, \ldots, x_r]$ for some $r \geq 1$. Let $X = \mathrm{Proj}S$ (projective r-space over A). Then* (1.4) *defines an isomorphism $\Gamma_*(\mathcal{O}_X) \cong S$.*

Proof. Cover X by the open subsets $D_+(x_i) \subset X$. By the sheaf axiom for $\mathcal{O}_X(n)$, we get an exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X(n)) \to \oplus_{i=0}^r (S_{x_i})_n \to \bigoplus_{i,j} (S_{x_i x_j})_n.$$

Taking the direct sum over all $n \in \mathbb{Z}$, we get an exact sequence

$$0 \to \Gamma_*(\mathcal{O}_X) \to \bigoplus_{i=0}^r S_{x_i} \to \bigoplus_{i,j} S_{x_i x_j}.$$

As the $x_i \in S$ are non-zero divisors, the maps

$$S \to S_{x_i} \to S_{x_i x_j} \to S' := S_{x_0 \cdots x_r}$$

are all injective. We get

$$\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^r S_{x_i} = S,$$

as subrings of S'. □

Exercise 1.2.4. More generally, let S be a graded ring finitely generated over S_0 by non-zero divisors $x_0, \ldots, x_r \in S_1$. Let $X = \mathrm{Proj}(S)$. Suppose that the $x_0, \ldots, x_r$ are relatively prime. Show that $S = \Gamma_*(\mathcal{O}_X)$.

Corollary 1.2.5. *(1) Let $X = \mathbb{P}_k^r = \mathrm{Proj}(k[x_0, \ldots, x_n])$. Then*

$$\Gamma(X, \mathcal{O}_X(n)) = (k[x_0, \ldots, x_r])_n.$$

In particular,

$$\Gamma(X, \mathcal{O}_X(1)) = (k[x_0, \ldots, x_r])_1 = \bigoplus_{i=0}^r k \cdot x_i.$$

(2) Let $X = \mathrm{Proj}(S)$ where S satisfies the assumptions in Exercise 1.2.4. Then $S_1 = \Gamma(X, \mathcal{O}_X(1))$. □

Definition 1.2.6. Let A be a ring and $r \geq 0$. We let $x_0, \ldots, x_r \in \mathcal{O}_{\mathbb{P}_A^r}(1)$ be the above global sections.

Lemma 1.2.7. *Let S be a graded ring, generated by S_1 as an S_0-module. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X = \mathrm{Proj}(S)$. Let $f \in S_1$. There are canonical isomorphisms*

$$\mathcal{F}(d)|_{D_+(f)} \cong f^d \cdot \mathcal{F}|_{D_+(f)}. \tag{1.5}$$

Proof. As $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(d)$, it suffices to prove the result for $\mathcal{F} = \mathcal{O}_X$. Notice that

$$S(d)_{(f)} = (S(d)_f)_0 = (S_f(d))_0 \,,$$

that

$$S_f(d) = \bigoplus_{e \in \mathbb{Z}} (S_f)_{d+e}, \quad (S_f)_{d+e} = \left\{ \frac{x}{f^m} \mid x \in S_{m+d+e} \right\},$$

and that the map

$$\begin{aligned} S_{(f)} &\longrightarrow (S_f(d))_0 = (S_f)_d = \left\{ \frac{x}{f^m} \mid x \in S_{m+d} \right\}, \\ \frac{y}{f^m} &\mapsto \frac{f^d \cdot y}{f^m} \in (S_f)_d \end{aligned}$$

is an isomorphism. More precisely, we have

$$f^d \cdot S_{(f)} = (S_f)_d \subset S_f.$$

Therefore, we have

$$\mathcal{O}_X(d)|_{D_+(f)} = \widetilde{S(d)_{(f)}} = \widetilde{(S(d)_f)_0} = \widetilde{(S_f)(d)_0} = \widetilde{f^d \cdot S_{(f)}} = f^d \cdot \widetilde{S_{(f)}} = f^d \cdot \mathcal{O}_X|_{D_+(f)}.$$

This proves the lemma. $\square$

Proposition 1.2.8. *Let S be a graded ring such that S is generated by S_1 as an S_0-algebra. Let $X = \mathrm{Proj}(S)$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then there is a natural isomorphism*

$$\psi \colon \widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}. \tag{1.6}$$

Proof. Let $f \in S_1$ and consider the scheme $D_+(f) = \mathrm{Spec}\, S_{(f)}$. We have

$$\Gamma(D_+(f), \widetilde{\Gamma_*(\mathcal{F})}) = (\Gamma_*(\mathcal{F}))_{(f)} = \left(\left(\bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{F}(d)) \right)_f \right)_0$$

This is an $S_{(f)}$-module; an element of this module is given by an expression

$$x = \frac{s}{f^d}, \qquad s \in \Gamma(X, \mathcal{F}(d)).$$

The canonical isomorphism (1.5) shows that the section

$$s|_{D_+(f)} \in \Gamma(D_+(f), \mathcal{F}(d))$$

is of the form

$$s|_{D_+(f)} = f^d \cdot t \qquad \text{for some} \qquad t \in \Gamma(D_+(f), \mathcal{F}).$$

We define $\varphi_f(x) \coloneqq t$, which gives a map

$$\varphi_f \colon \Gamma(D_+(f), \widetilde{\Gamma_*(\mathcal{F})}) \longrightarrow \Gamma(D_+(f), \mathcal{F}).$$

Since $D_+(f)$ is affine, and $\widetilde{\Gamma_*(\mathcal{F})}$ and $\mathcal{F}$ quasi-coherent, this yields a map

$$\psi_f \colon \widetilde{\Gamma_*(\mathcal{F})}|_{D_+(f)} \longrightarrow \mathcal{F}|_{D_+(f)}.$$

It is straightforward to show that the maps ψ_f and ψ_g agree on overlaps $D_+(f \cdot g) = D_+(f) \cap D_+(g)$, hence glue to give the morphism (1.6). It is also readily checked that ψ_f is an isomorphism for each $f \in S_1$. The result follows. $\square$

Lemma 1.2.9. *We have two functors*

$$F = (-)^{\sim} \colon \mathsf{GrMod}_S \longrightarrow \mathsf{QCoh}(X),$$
$$G = \Gamma_* \colon \mathsf{QCoh}(X) \longrightarrow \mathsf{GrMod}_S,$$

with $F \circ G \cong \mathrm{id}$ *as functors* $\mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$.

(1) The functor G is fully faithful, and that the functor F is essentially surjective.

(2) We do not in general have an isomorphism of functors $G \circ F \cong \mathrm{id}$.

Proof. (1). Essential surjectivity of F is clear: any object $\mathcal{M} \in \mathsf{QCoh}(X)$ is isomorphic to $(F \circ G)(\mathcal{M}) = F(G(\mathcal{M}))$. As for the faithfulness of G: this holds, as we have maps

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Hom}(G(\mathcal{M}), G(\mathcal{N})) \longrightarrow \mathrm{Hom}(FG(\mathcal{M}), FG(\mathcal{N})) \cong \mathrm{Hom}(\mathcal{M}, \mathcal{N})$$

whose composition is the identity. Hence the first map in the composition is injective.

(2). We give an example of a graded module M with $\Gamma_*(\widetilde{M}) \ncong M$. Let M be any non-zero graded S-module such that $\mathrm{Supp}(M) \subset V(S_+)$. Then $\widetilde{M} = 0$ hence $\Gamma_*(\widetilde{M}) = 0$. This finishes the proof. $\square$

1.2.2 Projective schemes

Definition 1.2.10. Let A be a ring. Let X be a scheme and let

$$f \colon X \to \mathrm{Spec}\ A$$

be a morphism of schemes. We say that f is *projective* if f admits a factorization

$$\begin{tikzcd} & \mathbb{P}^n_A \arrow[dr] & \\ X \arrow[ur, hook] \arrow[rr, "f"] & & \mathrm{Spec}\ A \end{tikzcd}$$

into a closed immersion $X \hookrightarrow \mathbb{P}^n_A$ and the canonical morphism $\mathbb{P}^n_A \to \mathrm{Spec}\ A$, for some integer $n \in \mathbb{Z}_{\geq 0}$. We also say that X is a *projective scheme* over A.

Comparison with the literature 1.2.11. See [Har77, Chapter II, Section 4.2, page 103]. See also [Liu02, Chapter 3, Section 3.1, Definition 1.12, page 83].

Lemma 1.2.12. *Let S be a graded ring. Let S' be another graded ring, and $\varphi\colon S \to S'$ is a surjective morphism of graded rings.*

(1) We have $S_+ \not\subset \varphi^{-1}(\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Proj}(S')$. In particular, $\mathrm{Bs}(\varphi) = \emptyset$, and we get a morphism of schemes $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$.

(2) The above morphism of schemes $\mathrm{Proj}(S') \to \mathrm{Proj}(S)$ is a closed immersion.

Proof. As for part (1), note that for $\mathfrak{p} \in \mathrm{Spec}\ S'$ homogeneous, we have

$$S'_+ \subset \mathfrak{p} \iff \varphi^{-1}(S'_+) \subset \varphi^{-1}(\mathfrak{p}) \iff S_+ \subset \varphi^{-1}(\mathfrak{p}),$$

where we use the fact that φ is surjective.

As for part (2), note that the morphism is locally given by the maps

$$\mathrm{Spec}\ (S'_{(\varphi(f))}) \to \mathrm{Spec}\ (S_{(f)}), \qquad f \in S.$$

These are induced by the ring maps

$$S_{(f)} \longrightarrow S'_{(\varphi(f))}. \tag{1.7}$$

In turn, the latter is induced via restriction by

$$S_f \longrightarrow S'_{\varphi(f)}.$$

This map is surjective: let $x/\varphi(f)^n \in S'_{\varphi(f)}$; then we can find $y \in S$ with $\varphi(y) = x$, so that

$$\varphi(y/f^n) = \varphi(y)/\varphi(f)^n \in S'_{\varphi(f)}.$$

Hence (1.7) is surjective (see Lemma (1.1.10)), proving (2). □

Proposition 1.2.13. *Let A be a ring.*

(1) Let X be a closed subscheme of $\mathbb{P}^r_A$. Then there exists a homogeneous ideal $I \subset A[x_0, \ldots, x_r]$ such that X is the closed subscheme determined by the surjective morphism of graded rings $A[x_0, \ldots, x_r] \to A[x_0, \ldots, x_r]/I$.

(2) A scheme X over $\mathrm{Spec}\ A$ is projective if and only if $X \cong \mathrm{Proj}(S)$ for some graded ring S such that $A = S_0$ and S is finitely generated by S_1 as an S_0-algebra.

Proof. (1). Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r_A}$ be the corresponding quasi-coherent ideal sheaf. By Proposition 1.2.8, there is a canonical isomorphism of graded S-modules

$$\widetilde{\Gamma_*(\mathcal{I})} \cong \mathcal{I}.$$

Moreover, the map

$$\Gamma_*(\mathcal{I}) \to \Gamma_*(\mathcal{O}_{\mathbb{P}^r_A})$$

is injective and identifies $\Gamma_*(\mathcal{I})$ with an ideal

$$I \subset \Gamma_*(\mathcal{O}_{\mathbb{P}^r_A}) = A[x_0, \dots, x_r],$$

where the canonical isomorphism $\Gamma_*(\mathcal{O}_{\mathbb{P}^r_A}) = A[x_0, \dots, x_r]$ was provided in Proposition 1.2.3. Hence we have

$$\mathcal{I} = \widetilde{I} \subset \widetilde{R} = \mathcal{O}_{\mathbb{P}^r_A}, \qquad R := A[x_0, \dots, x_r].$$

Item (1) follows from this.

(2). Suppose that X is projective. Then there is a closed immersion $X \hookrightarrow \mathbb{P}^r_A$ of schemes over A, for some $r \geq 0$. By item (1), we get that $X \cong \mathrm{Proj}(A[x_0, \dots, x_r]/I)$ for some homogeneous ideal $I \subset A[x_0, \dots, x_r]$. Conversely, if $X = \mathrm{Proj}(S)$ for some graded ring S with $A = S_0$ and S finitely generated by S_1 as S_0-algebra, then we can find elements $y_0, \dots, y_r \in S_1$ that generate S as an A-algebra. This gives a surjective morphism of graded A-algebras

$$A[x_0, \dots, x_r] \longrightarrow S, \qquad x_i \mapsto y_i,$$

yielding a closed immersion $\mathrm{Proj}(S) \hookrightarrow \mathbb{P}^r_A$ of schemes over A. □

1.2.3 Morphisms to projective space

Definition 1.2.14. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module for a scheme X. We say $\mathcal{F}$ is *generated by global sections* if there is an index set I and a surjective map of $\mathcal{O}_X$-modules

$$\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}.$$

Note that to give such a morphism is to give global sections $s_i \in \mathcal{F}$ for $i \in I$. We say that $\mathcal{F}$ is *globally generated* by the sections s_i.

Exercise 1.2.15. Let $S = k[u^4, u^3v, uv^3, v^4] \subset k[u, v]$, where the generators of S are considered as to have degree one (i.e. $\deg(u^4) = 1, \deg(u^3v) = 1$, etc.). Note that $\dim S_1 = 4$. Show that $\dim \Gamma(X, \mathcal{O}_X(1)) = 5$. Conclude that the canonical map $S_1 \to \Gamma(X, \mathcal{O}_X(1))$ is not surjective.

Example 1.2.16. (1) Let A be a ring, $X = \mathrm{Spec}\, A$, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F} \cong \widetilde{M}$ for some A-module M, and any set of generators for $M \cong \Gamma(X, \mathcal{F})$ will generate $\mathcal{F}$.

(2) Let S be a graded ring generated over S_0 by a subset $I \subset S_1$. Then the map

$$\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)$$

induced by the map $\beta\colon S_1 \to \Gamma(X, \mathcal{O}_X(1))$, is surjective.

Proof. Exercise. As for (2), suppose for instance that $S = A[x_0, \ldots, x_r]$, with $S_0 = A$. Then for each x_i, we have that

$$S(1)_{(f)} = A[x_0, \ldots, x_r](1)_{(x_i)} = (A[x_0, \ldots, x_r]_{x_i})_1$$

is generated by the x_i as an $A[x_0, \ldots, x_r]_{(x_i)}$-module. In fact, the map

$$S_{(x_i)} \longrightarrow S(1)_{(x_i)} = (S_{x_i})_1, \qquad s \mapsto x_i \cdot s$$

is an isomorphism of $S_{(x_i)}$-modules, with inverse $t \mapsto x_i^{-1} \cdot t$. Therefore, for each $i \in \{0, \ldots, r\}$, the images of the elements $x_0, \ldots, x_r \in S_1$ in $S(1)_{x_i} = (S_{x_i})_1$ generate $S(1)_{x_i}$ as an $S_{(x_i)}$-module. Thus, the map

$$\bigoplus_{i=0}^{r} S \longrightarrow S(1), \qquad (0, \ldots, 1, \ldots, 0) \mapsto x_i,$$

yields a surjection $\bigoplus_{i=0}^{r} \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)$. □

Lemma 1.2.17. *Let A be a ring, let $r \in \mathbb{Z}_{\geq 0}$ and consider a morphism of A-schemes $\varphi\colon X \to \mathbb{P}^r_A$. Then the global sections $x_0, \ldots, x_r \in \mathcal{O}_{\mathbb{P}^r_A}(1)$, see Definition 1.2.6, give rise to global sections*

$$s_i = \varphi^*(x_i) \in L := \varphi^*(\mathcal{O}_{\mathbb{P}^r_A}(1)), \qquad i = 0, \ldots, r,$$

that satisfy the property that L is globally generated by the sections s_i.

The following result shows that the converse is also true. An *isomorphism* between pairs $(L, (s_i))$ and $(M, (t_i))$, where L and M are line bundles on a scheme X and $s_0, \ldots, s_r, t_0, \ldots, t_r$ global sections, is an isomorphism $f\colon L \to M$ such that $s_i = f^*(t_i)$.

Theorem 1.2.18. *Let A be a ring. Let X be a scheme over A, and let L be a line bundle globally generated by sections $s_0, \ldots, s_r \in L$. Then there is a unique morphism*

$$\varphi\colon X \longrightarrow \mathbb{P}^r_A$$

such that

$$(\varphi^*(\mathcal{O}(1)), \varphi^*(x_0), \ldots, \varphi^*(x_r)) \cong (L, s_0, \ldots, s_r).$$

Proof. We do not prove this here. See e.g. [GW20, Corollary 13.33]. □

Corollary 1.2.19. *Let A be a ring. Consider the functor*

$$\begin{aligned} F\colon \mathsf{Sch}/A &\longrightarrow \mathsf{Set}, \\ X &\mapsto \{(L, s_0, \ldots, s_r) \mid L \text{ line bundle globally generated by the } s_i\} / \cong . \end{aligned}$$

This functor is representable by $\mathbb{P}^r_A$. More precisely, the association

$$\varphi \mapsto \left(\varphi^*(\mathcal{O}_{\mathbb{P}^r_A}(1)), \varphi^*(x_0), \ldots, \varphi^*(x_r)\right)$$

defines a bijection

$$\operatorname{Hom}_{\mathsf{Sch}/A}(X, \mathbb{P}^r_A) \xrightarrow{\sim} F(X)$$

for each A-scheme X, compatible with morphisms of A-schemes $X \to Y$.

For schemes X and T over $\mathbb{C}$, we define $X(T) \coloneqq \mathrm{Hom}_{\mathsf{Sch}/\mathbb{C}}(T, X)$ as the set of morphisms $T \to X$ of schemes over $\mathbb{C}$.

Example 1.2.20. We make the following observations and definitions:

(1) For a finite dimensional complex vector space V, we get a graded ring

$$S = \mathrm{Sym}^*(V) = \oplus_{d \geq 0} \mathrm{Sym}^d(V)$$

with $S_0 = \mathbb{C}$. If we choose a basis $\{e_0, \ldots, e_r\}$ for V, then each $e_i \in V$ defines an element $x_i \in \mathrm{Sym}^1(V)$ so that we get a set $\{x_0, \ldots, x_r\} \subset S_1 = \mathrm{Sym}^1(V) = V$ of generators for S as an $S_0 = \mathbb{C}$-algebra, in a way that $S = \mathbb{C}[x_0, \ldots, x_r]$.

(2) We define

$$\mathbb{P}(V) \coloneqq \mathrm{Proj}(\mathrm{Sym}^*(V)), \qquad \check{\mathbb{P}}(V) \coloneqq \mathrm{Proj}(\mathrm{Sym}^*(V^\vee)).$$

This gives back $\mathbb{P}^r_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^{r+1})$.

(3) We define

$$\check{\mathbb{P}}^r_{\mathbb{C}} \coloneqq \mathbb{P}((\mathbb{C}^{r+1})^\vee).$$

(4) Using Corollary 1.2.19, we can show that there is a canonical bijection

$$\check{\mathbb{P}}(V)(\mathbb{C}) = \{\text{lines } \ell \subset V\}.$$

In particular, we get a canonical bijection

$$\check{\mathbb{P}}^r_{\mathbb{C}}(\mathbb{C}) = \left\{\text{lines } \ell \subset \mathbb{C}^{r+1}\right\}.$$

(5) Via the canonical identification $\mathbb{C}^{r+1} \cong (\mathbb{C}^{r+1})^\vee$ that sends e_i to $e_i^\vee$, this gives

$$\mathbb{P}^r_{\mathbb{C}}(\mathbb{C}) = \check{\mathbb{P}}^r_{\mathbb{C}}(\mathbb{C}) = \left\{\text{lines } \ell \subset \mathbb{C}^{r+1}\right\}.$$

In other words, we re-obtain the good old description of the projective space of dimension n as the space of lines in the affine space of dimension $n + 1$.

Proof. Exercise. □

Example 1.2.21. Let k be a field. Let $L \supset k$ be a field extension of k. Let M be a line bundle on $X \coloneqq \mathrm{Spec}\ L$. Then $M \cong \mathcal{O}_X$. Therefore,

$$\mathbb{P}^n_k(L) = \mathrm{Hom}_{\mathsf{Sch}/k}(\mathrm{Spec}\ L, \mathbb{P}^n_k) = \left\{(s_0, \ldots, s_n) \in L^{n+1} - \{0\}\right\} /_\sim$$

where $(s_0, \ldots, s_n) \sim (t_0, \ldots, t_n)$ if there exists $\lambda \in L^*$ with $\lambda \cdot (s_0, \ldots, s_n) = (t_0, \ldots, t_n)$.

Example 1.2.22. Let k be a field and let R be a k-algebra which is a discrete valuation ring. Let $X = \mathrm{Spec}\ R$. We will prove later (see Propositions 3.2.17 and 3.4.8) that any line bundle L on X is isomorphic to $\mathcal{O}_X$. Let $\mathfrak{m} \subset R$ be the maximal ideal. Then

$$\mathbb{P}^n_k(R) = \mathrm{Hom}_{\mathsf{Sch}/k}(X, \mathbb{P}^n_k) = \left\{(s_0, \ldots, s_n) \in (R^{n+1} - \{0\}) \mid \exists i \mid s_i(\mathfrak{m}) \neq 0\right\} /_\sim$$

where $(s_0, \ldots, s_n) \sim (t_0, \ldots, t_n)$ if there exists $\lambda \in R^*$ with $\lambda \cdot (s_0, \ldots, s_n) = (t_0, \ldots, t_n)$.

Chapter 2

Cohomology

In this chapter, we consider an abelian sheaf $\mathcal{F}$ on a scheme X, and define cohomology groups $\mathrm{H}^i(X,\mathcal{F})$ for $i \in \mathbb{Z}_{\geq 0}$. They have the property that if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of abelian sheaves, then one gets a long exact sequence:

$$0 \to \Gamma(X,\mathcal{F}_1) \to \Gamma(X,\mathcal{F}_2) \to \Gamma(X,\mathcal{F}_3) \to \mathrm{H}^1(X,\mathcal{F}_1) \to \mathrm{H}^1(X,\mathcal{F}_2) \to \cdots .$$

Thus, the cohomology measures the failure of the right exactness of the global sections functor $\Gamma(X,-)$. Moreover, if $(X_i,\mathcal{F}_i)$ $(i = 1,2)$ are schemes with sheaves on them, and if $\phi\colon X_1 \to X_2$ is an isomorphism with $\phi^{-1}\mathcal{F}_2 \cong \mathcal{F}_1$, then one has an isomorphism $\mathrm{H}^p(X_1,\mathcal{F}_1) \cong \mathrm{H}^p(X_2,\mathcal{F}_2)$ for each $p \geq 0$. Thus, sheaf cohomology forms an invariant of the pair $(X,\mathcal{F})$. This invariant turns out to be important in many situations.

2.1 Lecture 16 : Cech cohomology of sheaves on a scheme

2.1.1 Some homological algebra

Definition 2.1.1. A *complex of abelian groups* $A^\bullet$ is a sequence of groups A^i indexed by $\mathbb{Z}$ together with maps d_A^i between them as follows:

$$\cdots \xrightarrow{d_A^{i-2}} A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \xrightarrow{d_A^{i+1}} \cdots ,$$

such that $d_A^i \circ d_A^{i-1} = 0$. A *morphism* of complexes

$$f^\bullet\colon A^\bullet \to B^\bullet$$

is a collection of maps $f_p\colon A^p \to B^p$ such that $f_i \circ d_A^{i-1} = d_B^{i-1} \circ f_{i-1}$ for each $i \in \mathbb{Z}$. In this way, we can talk about *kernels, images, cokernels* and *exact sequences* of complexes of abelian groups. We define

$$\mathrm{H}^p(A^\bullet) := \mathrm{Ker}(d_A^p)/\mathrm{Im}(d_A^{p-1}).$$

Lemma 2.1.2. *Let $0 \to F^\bullet \to G^\bullet \to H^\bullet \to 0$ be an exact sequence of complexes of abelian groups. Then there is an associated long exact sequence of cohomology groups*

$$\cdots \to H^p(F^\bullet) \to H^p(G^\bullet) \to H^p(H^\bullet) \to H^{p+1}(F^\bullet) \to \cdots .$$

Proof. We have a commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & F^p \arrow[r] \arrow[d] & G^p \arrow[r] \arrow[d] & H^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1} \arrow[r] & G^{p+1} \arrow[r] & H^{p+1} \arrow[r] & 0
\end{tikzcd}$$

By the Snake lemma, we get an exact sequence

$$0 \to \mathrm{Ker}(d_F^p) \to \mathrm{Ker}(d_G^p) \to \mathrm{Ker}(d_H^p) \to F^{p+1}/\mathrm{Im}(d_F^p) \to \cdots .$$

Consider now the diagram

$$\begin{tikzcd}
 & F^p/\mathrm{Im}(d^{p-1}) \arrow[r] \arrow[d, "d"] & G^p/\mathrm{Im}(d^{p-1}) \arrow[r] \arrow[d] & H^p/\mathrm{Im}(d_H^p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d^{p+1}) \arrow[r] & \mathrm{Ker}(d^{p+1}) \arrow[r] & \mathrm{Ker}(d_H^{p+1}).
\end{tikzcd}$$

It has exact rows by the previous argument. Applying the Snake lemma again, gives an exact sequence

$$H^p(F^\bullet) \to H^p(G^\bullet) \to H^p(H^\bullet) \to H^{p+1}(F^\bullet) \to H^{p+1}(G^\bullet) \to H^{p+1}(H^\bullet).$$

Since this sequence is exact for every $p \in \mathbb{Z}$, the result follows. $\square$

Let $f\colon C^\bullet \to D^\bullet$ be a morphism of complexes $C^\bullet$ and $D^\bullet$. Then, since $f \circ d_C = d_D \circ f$, the map f induces a well-defined map on cohomology groups

$$f\colon H^i(C^\bullet) \to H^i(D^\bullet).$$

Definition 2.1.3. A *chain homotopy* between two morphisms $f, g\colon C^\bullet \to D^\bullet$ is a collection of maps $h\colon C^n \to D^{n-1}$ such that

$$f - g = d_D \circ h + h \circ d_C.$$

Lemma 2.1.4. *If there exists a chain homotopy between f and g, then f and g induce the same map $H^i(C^\bullet) \to H^i(D^\bullet)$.*

Proof. Let $c \in \mathrm{Ker}(C^i \to C^{i+1})$. Then $[f(c) - g(c)] = [d_D(h(c))] = 0 \in H^i(D^\bullet)$. $\square$

Exercise 2.1.5. Let $C^\bullet$ be a complex.

(1) Show that $C^\bullet$ is exact if and only if $H^i(C^\bullet) = 0$ for all i.

(2) Assume that there exists a chain homotopy $h\colon C^n \to C^{n-1}$ between the identity $\mathrm{id}\colon C^\bullet \to C^\bullet$ and the zero map $0\colon C^\bullet \to C^\bullet$. Show that $c = d^p \circ h(c) + h \circ d(c)$ for every $c \in C^{p+1}$. Show that $H^i(C^\bullet) = 0$ for each i, hence that $C^\bullet$ is exact.

2.1.2 Cech cohomology

Let X be a topological space. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X, indexed by some set I. By the well-ordering theorem, there exists a well-ordering I, which we choose once and for all. For any finite set of indices $i_0, \ldots, i_p \in I$, we denote

$$U_{i_0,\ldots,i_p} \coloneqq U_{i_0} \cap \cdots \cap U_{i_p}.$$

For a sheaf $\mathcal{F}$ on X, we have the sheaf sequence

$$0 \to \mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j).$$

Definition 2.1.6. Let X and $\mathcal{U}$ be as above. Let $\mathcal{F}$ be a sheaf on X. We define the *Cech complex* of $\mathcal{F}$ (with respect to $\mathcal{U}$) as the complex $C^\bullet(\mathcal{U}, \mathcal{F})$ with

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,u_p}).$$

Thus, to given an element $\alpha \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is to give a $(p+1)$-tuple of elements

$$\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots,i_p})$$

for *each* strictly increasing $(p+1)$-tuple $i_0 < \cdots < i_p$ of elements of I. We define the coboundary map $d^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ as the map that sends $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ to the element $d\alpha \in C^{p+1}(\mathcal{U}, \mathcal{F})$ with

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} \in \mathcal{F}(U_{i_0,\ldots,i_{p+1}}).$$

Here, the notation $\widehat{i_k}$ means that we omit i_k.

Let $\alpha \in C^0(\mathcal{U}, \mathcal{F}) = \prod_{i \in I} \mathcal{F}(U_i)$. Then

$$(d\alpha)_{i_0,i_1} = \alpha_{i_1}|_{U_{i_0,i_1}} - \alpha_{i_0}|_{U_{i_0,i_1}} \in \mathcal{F}(U_{i_0,i_1}).$$

Hence, for each $i_0, i_1, i_2 \in I$ with $i_0 < i_1 < i_2$, we have:

$$\begin{aligned}
(d^2\alpha)_{i_0,i_1,i_2} &= (d\alpha)_{i_1,i_2}|_{U_{i_0,i_1,i_2}} - (d\alpha)_{i_0,i_2}|_{U_{i_0,i_1,i_2}} + (d\alpha)_{i_0,i_1}|_{U_{i_0,i_1,i_2}} \\
&= \Big(\big(\alpha_{i_2}|_{U_{i_1,i_2}} - \alpha_{i_1}|_{U_{i_1,i_2}}\big) - \big(\alpha_{i_2}|_{U_{i_0,i_2}} - \alpha_{i_0}|_{U_{i_0,i_2}}\big) + \big(\alpha_{i_1}|_{U_{i_0,i_1}} - \alpha_{i_0}|_{U_{i_0,i_1}}\big) \Big) |_{U_{i_0,i_1,i_2}} \\
&= \big(\alpha_{i_2}|_{U_{i_0,i_1,i_2}} - \alpha_{i_1}|_{U_{i_0,i_1,i_2}}\big) - \big(\alpha_{i_2}|_{U_{i_0,i_1,i_2}} - \alpha_{i_0}|_{U_{i_0,i_1,i_2}}\big) + \big(\alpha_{i_1}|_{U_{i_0,i_1,i_2}} - \alpha_{i_0}|_{U_{i_0,i_1,i_2}}\big) \\
&= 0.
\end{aligned}$$

In particular, we get $d \circ d = 0$ as maps $C^0(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$. This generalizes as follows.

Lemma 2.1.7. *We have $d^{p+1} \circ d^p = 0$ as maps $C^p(\mathcal{U}, \mathcal{F}) \to C^{p+2}(\mathcal{U}, \mathcal{F})$.*

Proof. Exercise. □

Definition 2.1.8. The *p-th Cech cohomology group* of $\mathcal{F}$ with respect to $\mathcal{U}$ is the group

$$\mathrm{H}^p(\mathcal{U},\mathcal{F}) \coloneqq \mathrm{H}^p(C^\bullet(\mathcal{U},\mathcal{F})) = \mathrm{Ker}(d^p)/\mathrm{Im}(d^{p-1}).$$

Notice that a sheaf homomorphism $\mathcal{F} \to \mathcal{G}$ induces morphisms $C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{G})$, and it is not hard to show that these induce morphisms

$$\mathrm{H}^p(\mathcal{U},\mathcal{F}) \to \mathrm{H}^p(\mathcal{U},\mathcal{G}).$$

This gives functors $\mathrm{H}^p(\mathcal{U},-)$ from abelian sheaves on X to abelian groups.

Example 2.1.9. Notice that

$$\mathrm{H}^0(\mathcal{U},\mathcal{F}) = \mathrm{Ker}\left(\prod_i \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_i \cap U_j)\right) = \mathcal{F}(X).$$

Example 2.1.10. The group $\mathrm{H}^1(\mathcal{U},\mathcal{F})$ is the group of sections $\sigma_{ij} \in \prod_{i<j} \mathcal{F}(U_{ij})$ such that $\sigma_{ik}|_{U_{ijk}} = \sigma_{ij}|_{U_{ijk}} + \sigma_{jk}|_{U_{ijk}}$, modulo the sections σ_{ij} of the form $\sigma_{ij} = \tau_j|_{U_{ij}} - \tau_i|_{U_{ij}}$.

Example 2.1.11. Consider a short exact sequence of abelian sheaves on X:

$$0 \to \mathcal{A} \to \mathcal{B} \xrightarrow{f} \mathcal{C} \to 0.$$

Let $c \in \mathcal{C}(X)$. Let $\mathcal{U} = \{U_i\}_i$ be an open covering of X such that $c|_{U_i} = f(b_i)$ for some $b_i \in \mathcal{B}(U_i)$. Define

$$\sigma_{ij} \coloneqq b_j|_{U_{ij}} - b_i|_{U_{ij}} \in \mathcal{A}(U_{ij}).$$

(1) We have $\sigma_{ik}|_{U_{ijk}} = \sigma_{ij}|_{U_{ijk}} + \sigma_{jk}|_{U_{ijk}}$.

(2) Let

$$\sigma(c) \in \mathrm{H}^1(\mathcal{U},\mathcal{A})$$

be the Cech cohomology class induced by the c_{ij}. Then $\sigma(c) = 0$ if and only if there exists an element $b \in \mathcal{B}(X)$ with $f(b) = c$.

Definition 2.1.12. Let P be a property that a morphism of schemes can have. For instance, P can be being a closed immersion, an open immersion, surjective, an isomorphism, etc. We say that the property P is *stable under base change* if for any morphism of schemes $X \to Y$ that has property P, any scheme T and any morphism of schemes $T \to Y$, the resulting morphism of schemes $X \times_Y T \to T$ has property P.

Lemma 2.1.13. *The property of being a closed immersion is stable under base change.*

Proof. Let $f\colon X \to Y$ be a closed immersion. We consider a morphism of schemes $T \to Y$; the goal is to show that $\pi\colon X \times_Y T \to T$ is a closed immersion. It suffices to provide an affine open covering $\{T_i\}$ of T such that $\pi^{-1}(T_i)$ is affine and $\pi^{-1}(T_i) \to T_i$ is a closed immersion. We start with an affine open covering $\{Y_i\}$ of Y, which gives an open covering of T (by taking inverse images under $T \to Y$) which we refine to an affine open covering $\{T_j\}$ of T. Thus, for each $j \in J$ there is an $i \in I$ such that T_j maps into Y_i under $T \to Y$. Then $\pi^{-1}(T_j) = f^{-1}(Y_i) \times_{Y_i} T_j$ is affine, and the map $\mathcal{O}(T_j) \to \mathcal{O}(f^{-1}(Y_i)) \otimes_{\mathcal{O}(Y_i)} \mathcal{O}(T_j)$ is surjective as $\mathcal{O}(Y_i) \to \mathcal{O}(f^{-1}(Y_i))$ is surjective. □

Lemma 2.1.14. *Let X be a separated scheme. Let $U \subset X$ and $V \subset X$ be affine opens. Then $U \cap V$ is affine.*

Proof. Notice that $U \cap V = U \times_X V$. This is naturally a closed subscheme of $U \times_{\mathbb{Z}} V$, since it sits inside the cartesian diagram

$$\begin{tikzcd} U \times_X V \arrow[r, hook] \arrow[d] & U \times_{\mathbb{Z}} V \arrow[d] \\ X \arrow[r, hook] & X \times_{\mathbb{Z}} X, \end{tikzcd}$$

and closed immersions are stable under base change by Lemma 2.1.13. Moreover, $U \times_{\mathbb{Z}} V = U \times_{\mathrm{Spec}\,(\mathbb{Z})} V$ is affine, because U, V and Spec $(\mathbb{Z})$ are all affine. As closed subschemes of affine schemes are affine, we are done. □

Theorem 2.1.15. *Let X be a noetherian separated scheme. Let $\mathcal{U} = \{U_0, U_1, \ldots, U_r\}$ be a finite covering of X by affine opens $U_i \subset X$. Then all the intersections $U_{i_0,\ldots,i_p}$ are affine, and moreover:*

(1) The Cech cohomology groups define functors $\mathrm{H}^i(\mathcal{U}, -)\colon \mathsf{AbSh}_X \to \mathsf{Ab}$.

(2) We have $\mathrm{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

(3) Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then there is an associated long exact sequence in cohomology:

$$\cdots \to \mathrm{H}^i(\mathcal{U}, \mathcal{F}_1) \to \mathrm{H}^i(\mathcal{U}, \mathcal{F}_2) \to \mathrm{H}^i(\mathcal{U}, \mathcal{F}_3) \to \mathrm{H}^{i+1}(\mathcal{U}, \mathcal{F}_1) \to \mathrm{H}^{i+1}(\mathcal{U}, \mathcal{F}_2) \to \cdots.$$

(4) If $\mathcal{V} = \{V_j\}$ is another finite covering of X by affine opens, then there is a canonical isomorphism

$$\mathrm{H}^p(\mathcal{U}, \mathcal{F}) = \mathrm{H}^p(\mathcal{V}, \mathcal{F})$$

for every $p \geq 0$ and every quasi-coherent sheaf $\mathcal{F}$ on X.

(5) If X has dimension n, then $\mathrm{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ on X and every integer $p > n$.

Proof. Finite intersections of affines on separated scheme are affine. Indeed, this follows from Lemma 2.1.14 above.

(1) & (2). We have already observed this above.

(3). Note that if $U \subset X$ is an affine open subset, then the sequence

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0$$

is exact, because the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ from quasi-coherent $\mathcal{O}_U$-modules to $\mathcal{O}_X(U)$-modules is exact as U is affine. It follows that for each $p \geq 0$ and each $i_0 < \cdots < i_p \in I$, the sequence

$$0 \to \mathcal{F}_1(U_{i_0,\ldots,i_p}) \to \mathcal{F}_2(U_{i_0,\ldots,i_p}) \to \mathcal{F}_3(U_{i_0,\ldots,i_p}) \to 0$$

is exact (again since $U_{i_0,\dots,i_p}$ is affine). Therefore, the sequence

$$0 \to C^p(\mathcal{U}, \mathcal{F}_1) \to C^p(\mathcal{U}, \mathcal{F}_2) \to C^p(\mathcal{U}, \mathcal{F}_3) \to 0$$

is exact for each $p \geq 0$, so that we get an exact sequence of complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}_1) \to C^\bullet(\mathcal{U}, \mathcal{F}_2) \to C^\bullet(\mathcal{U}, \mathcal{F}_3) \to 0.$$

Hence the desired long exact sequence comes from Lemma 2.1.2.

(4). We do not prove this here.

(5). We only prove this in case X is quasi-projective of finite type over a noetherian ring A. In this case, X admits an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ consisting of $m \leq n+1$ affine open subsets $U_i \subset X$, see Exercise 2.1.16 below. In particular, $C^p(\mathcal{U}, \mathcal{F}) = 0$ for $p \geq m$, since there are no $(p+1)$-tuples $i_0 < \cdots < i_p \in I$, for $p \geq m$. □

Exercise 2.1.16. Let X be a quasi-projective scheme of finite type over a noetherian ring A. Let $n = \dim(X)$. Then X admits an affine open cover $\mathcal{U}$ consisting of at most $n+1$ affine open subsets $U_i \subset X$.

Proof hint: Suppose that $X \subset Z \subset \mathbb{P}^r_A$, where Z is a closed subscheme of $\mathbb{P}^r_A$ and X is an open subscheme of Z. Write $W = Z - X$. Write $Z = \cup_i Z_i$ as a union of its irreducible components. If $Z_i \subset W$, then $X = Z - W \subset Z - Z_i$, so that $X \cap Z_i = \emptyset$, hence $X \subset \cup_{j\neq i} Z_j$. Therefore, one may assume that the irreducible components of Z are not contained in W. Using induction on the dimension, one can prove that X is covered by $n+1$ open affines induced from open affines in $\mathbb{P}^r_A$. □

We record here the following lemma, for later use:

Lemma 2.1.17. *Let $\pi\colon X \to Y$ be a morphism of noetherian separated schemes, such that the scheme $\pi^{-1}(U)$ is affine for every affine open $U \subset Y$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and let $i \geq 0$ be an integer. Then we have a canonical isomorpism*

$$\mathrm{H}^i(Y, \pi_*\mathcal{F}) = \mathrm{H}^i(X, \mathcal{F}).$$

Proof. Let $\mathcal{U} = \{U_i\}$ be a finite affine open covering of Y such that $\mathrm{H}^i(Y, \pi_*\mathcal{F})$ is computed by the Cech complex $C^\bullet(\mathcal{U}, \pi_*\mathcal{F})$. Then $\mathcal{V} := \{\pi^{-1}(U_i)\}$ forms an affine open covering of X, and we have a canonical isomorphism $C^\bullet(\mathcal{U}, \pi_*\mathcal{F}) = C^\bullet(\mathcal{V}, \mathcal{F})$. □

2.2 Lecture 17 : Examples & Cohomology via resolutions

2.2.1 Some examples

Proposition 2.2.1. *Let k be a field. We consider $\mathbb{P}^1 := \mathbb{P}^1_k = \mathrm{Proj}k[x_0, x_1]$. Then there is a natural isomorphism between $\mathbb{P}^1$ and the scheme obtained by glueing together $U_0 = \mathrm{Spec}\ k[t]$ and $U_1 = \mathrm{Spec}\ k[t^{-1}]$ along $\mathrm{Spec}\ k[t, t^{-1}]$.*

Proof. We have isomorphism

$$U_i := D_+(x_i) \cong \operatorname{Spec} k[x_0, x_1]_{(x_i)}$$

for $i = 0, 1$. Moreover, there is a map of k-algebras

$$\varphi_0 \colon k[t] \to k[x_0, x_1]_{(x_0)}, \quad t \mapsto \frac{x_1}{x_0}.$$

Then φ_0 is an isomorphism, with inverse $s \mapsto s(1, t)$. Similarly, we have

$$\varphi_1 \colon k[t^{-1}] \cong k[x_0, x_1]_{(x_1)}, \quad t^{-1} \mapsto \frac{x_0}{x_1}.$$

Finally, $D_+(x_0x_1) = \operatorname{Spec} k[x_0, x_1]_{(x_0x_1)}$, and there is an isomorphism $k[t, t^{-1}] \cong k[x_0, x_1]_{(x_0x_1)}$ defined as $t \mapsto x_0^2/(x_0x_1)$ and $t^{-1} \mapsto x_1^2/(x_0x_1)$. □

Example 2.2.2. Consider the projective line $\mathbb{P}^1 = \mathbb{P}^1_k$ as above; it is covered by the open affines $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$ with intersection $U_0 \cap U_1 = \operatorname{Spec} k[t, t^{-1}]$. Let $\mathcal{U} = \{U_0, U_1\}$. For the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, the Cech complex

$$0 \to C^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) \to C^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) \to 0$$

takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(U_0) \times \mathcal{O}_{\mathbb{P}^1}(U_1) \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(U_0 \cap U_1) \arrow[r] & 0 \\
0 \arrow[r] & k[t] \times k[t^{-1}] \arrow[r, "d"] \arrow[u] & k[t, t^{-1}] \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

with

$$d\big(f(t), g(t^{-1})\big) = g(t^{-1}) - f(t).$$

If $f(t) = g(t^{-1}) \in k[t, t^{-1}]$, then $f = g \in k$. In other words,

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathrm{H}^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Ker}(d) = k.$$

Furthermore, each element $s \in k[t, t^{-1}]$ is a sum of a polynomial in t and a polynomial in t^{-1}. Therefore, d is surjective, so that

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathrm{H}^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = 0.$$

Example 2.2.3. Let $m \in \mathbb{Z}$, consider $\mathbb{P}^1 := \mathbb{P}^1_k$, the projective line over a field k, and the sheaf $\mathcal{O}(m) := \mathcal{O}_{\mathbb{P}^1}(m)$. Let $S = k[x_0, x_1]$. We have

$$\mathcal{O}(m)(D_+(x_i)) = S(m)_{(x_i)} = x_i^m \cdot S_{(x_i)}$$

for $i = 1, 2$. Under the isomorphisms

$$\begin{aligned}
S_{(x_0)} &\to k[t], \quad f \mapsto f(1, t) \\
S_{(x_1)} &\to k[t^{-1}], \quad f \mapsto f(t^{-1}, 1), \\
S_{(x_0x_1)} &\to k[t, t^{-1}], \quad f \mapsto f(1, t) = f(t^{-1}, 1),
\end{aligned}$$

see Proposition 2.2.1, the Cech complex takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(m)(U_0) \times \mathcal{O}_{\mathbb{P}^1}(m)(U_1) \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(m)(U_0 \cap U_1) \arrow[r] & 0 \\
0 \arrow[r] & x_0^m \cdot S_{(x_0)} \times x_1^m \cdot S_{(x_1)} \arrow[r, "d"] \arrow[u] & x_1^m \cdot S_{(x_0 x_1)} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & S_{(x_0)} \times (\frac{x_1}{x_0})^m \cdot S_{(x_1)} \arrow[r] \arrow[u] & (\frac{x_1}{x_0})^m \cdot S_{(x_0 x_1)} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & k[t] \times t^m \cdot k[t^{-1}] \arrow[r] \arrow[u] & t^m \cdot k[t, t^{-1}] \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Here, we have

$$d\colon x_0^m \cdot S_{(x_0)} \times x_1^m \cdot S_{(x_1)} \longrightarrow x_1^m \cdot S_{(x_0x_1)}, \quad d\,(x_0^m \cdot f, x_1^m \cdot g) = x_1^m \cdot g - \frac{x_0^m}{x_1^m} \cdot x_1^m \cdot f,$$

corresponding to the map

$$d\colon k[t] \times t^m \cdot k[t^{-1}] \longrightarrow t^m \cdot k[t, t^{-1}] = k[t, t^{-1}], \quad d(f(t), t^m \cdot g(t^{-1})) \mapsto t^m \cdot g - f.$$

Suppose that $m \geq 0$. Then the elements

$$(t^m, t^m \cdot 1), (t^{m-1}, t^m \cdot t^{-1}), \ldots, (t^0, t^m \cdot t^{-m})$$

are linearly independent elements that generate the kernel of d. Therefore,

$$\dim \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1)) = \dim \mathrm{H}^0(\mathcal{U}, \mathcal{O}(1)) = \dim \operatorname{Ker}(d) = m + 1.$$

If $m < 0$, then $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1)) = 0$.

Example 2.2.4. Next, we compute the dimension of $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(m))$. If $m \geq 0$, then any polynomial in $k[t, t^{-1}]$ can be written in the form $t^m g(t^{-1}) - f(t)$ for $f(t) \in k[t]$ and $g(t^{-1}) \in k[t^{-1}]$. We claim the same holds if $m = -1$. Indeed, let $t^{-k} \in k[t, t^{-1}]$ for some $k \geq 1$ (for the non-negative powers of t, the claim is clear). Then $t^{-k} = t^{-1} \cdot t^{-k+1}$, with $t^{-(k-1)} \in k[t^{-1}]$ as $k - 1 \geq 0$. Therefore, the map

$$k[t] \times t^m \cdot k[t^{-1}] \to t^m \cdot k[t, t^{-1}], \qquad (f, t^m \cdot g) \mapsto t^m \cdot g - f$$

is surjective if $m \geq -1$. Hence $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(m)) = 0$ for $m \geq -1$.

If $m \leq -2$, then no linear combinations of the monomials

$$t^{-1}, t^{-2}, \ldots, t^{m+1} = t^{-(-m-1)}$$

lies in the image of d, but combinations of all the others do. It follows that $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(m))$ is a k-vector space of dimension $-m - 1$ in this case.

Example 2.2.5. We now consider an example from topology. Let $X = S^1$ be the unit circle, with the standard euclidean topology. Let $\mathcal{U} = \{U, V\}$, where U and V are connected open intervals that intersect in two connected open intervals W_1 and W_2. Let $\mathcal{F} = \mathbb{Z}_X$ be the constant sheaf associated to $\mathbb{Z}$. Then, we have

$$C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U) \times \mathcal{F}(V) = \mathbb{Z} \times \mathbb{Z}, \qquad C^1(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U \cap V) = \mathcal{F}(W_1 \sqcup W_2) = \mathbb{Z} \times \mathbb{Z}.$$

Under these identifications, the map $d\colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ is given by

$$d\colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}, \qquad (a, b) \mapsto (b, b) - (a, a) = (b - a, b - a).$$

Hence:

$$\mathrm{H}^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(d) = \mathrm{Im}(\mathbb{Z} \xrightarrow{x \mapsto (x,x)} \mathbb{Z} \times \mathbb{Z}) \cong \mathbb{Z}.$$

and

$$\mathrm{H}^1(\mathcal{U}, \mathcal{F}) = (\mathbb{Z} \times \mathbb{Z})\,/\mathrm{Im}(d) \cong \mathbb{Z}.$$

This gives the same answer as singular cohomology.

Remark 2.2.6. This is no coincidence: the groups $H^p(\mathcal{U}, \mathbb{Z})$ agree with the usual singular cohomology groups $H^p_{sing}(X, \mathbb{Z})$ for any topological space X homotopy equivalent to a CW complex, provided that the open sets in the covering $\mathcal{U}$ are contractible.

Exercise 2.2.7. Let X be a topological space and let $\mathcal{U}$ be an open cover of X. Assume that $U_i = X$ for some $i \in I$. Show that $H^p(\mathcal{U}, \mathcal{F}) = 0$ for every abelian sheaf $\mathcal{F}$ on X and every integer $p \geq 1$.

Example 2.2.8. Let X be an irreducible topological space. Then X is connected and any non-empty open subset $U \subset X$ is irreducible, hence connected. Let A_X be the constant sheaf associated to an abelian group A. Then $A_X(U) = A$ for any non-empty open $U \subset X$ (so that A_X agrees with the constant presheaf associated to A).

Let $\mathcal{U}$ be an open covering of X whose index set I is well-ordered. The Cech complex takes the form

$$0 \to \prod_{i_0 \in I} A \to \prod_{i_0 < i_1} A \to \prod_{i_0 < i_1 < i_2} A \to \cdots,$$

where for $\alpha \in \prod_{i_0 < \cdots < i_p} A$, we have its coordinate $\alpha_{i_0, \ldots, i_p} \in A$, and:

$$d(\alpha)_{i_0, \ldots, i_{p+1}} = \sum_{k=0,\ldots,p+1} (-1)^k \alpha_{i_0, \ldots, \widehat{i_k}, \ldots, i_p} \in A.$$

Note also that $H^p(\mathcal{U}, \mathcal{F}) = 0$ in view of Exercise 2.2.7. Indeed, by the above, the Cech complex does not depend on the U_i, only on the index set I. Hence we may assume $U_i = X$ for some i.

2.2.2 Cohomology as right derived functor

Definition 2.2.9. (1) Let A be an abelian group. Then A is *injective* if the contravariant functor $\mathrm{Hom}(-, A)$ from Ab to Ab, is exact. This is equivalent to saying that it is right exact. In other words, for any injective morphism $B_1 \hookrightarrow B_2$ of abelian groups, and any morphism $B_1 \to A$, there should exist a morphism $B_2 \to A$ that makes the obvious triangle commute.

(2) Let $\mathcal{F}$ be an abelian sheaf on a topological space X. Then $\mathcal{F}$ is *injective* if the contravariant functor $\mathrm{Hom}(-, \mathcal{F})$ from $\mathsf{AbSh}(X)$ to Ab, is exact. This is equivalent to saying that it is right exact. In other words, for any injective morphism $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ of abelian sheaves, and any morphism $\mathcal{B}_1 \to \mathcal{F}$, there should exist a morphism $\mathcal{B}_2 \to \mathcal{F}$ that makes the obvious triangle commute.

Exercise 2.2.10. (1) Show that an abelian group A is injective if and only if it is *divisible*: for each $n \in \mathbb{Z}_{\geq 1}$ and each $x \in A$ there exists $y \in A$ such that $n \cdot y = x$.

(2) Give an example of a non-zero divisible abelian group A such that for each $a \in A$ there exists $n \in \mathbb{Z}_{\geq 1}$ such that $n \cdot a = 0$.

(3) Show that a finite abelian group which is divisible, is zero.

(4) Show that the quotient of a divisible abelian group is divisible.

Proposition 2.2.11. *Let X be a topological space. Then any abelian sheaf $\mathcal{F}$ admits an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$ into an injective abelian sheaf $\mathcal{I}$.*

Proof. We first prove the proposition in the case where $X = \{x\}$ is a point. Then $\mathcal{F}$ corresponds to an abelian group A, and we need to find an injective morphism $A \hookrightarrow I$ into a divisible abelian group I (see the above exercise). Consider the morphism

$$F := \bigoplus_{a \in A} \mathbb{Z} \longrightarrow A, \qquad \sum_a n_a \mapsto \sum_a n_a \cdot a.$$

This is clearly a surjective group homomorphism. Let K be the kernel. There is an embedding

$$F \hookrightarrow F \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{a \in A} \mathbb{Q},$$

and hence an embedding

$$A = F/K \hookrightarrow (F \otimes_{\mathbb{Z}} \mathbb{Q})/K.$$

As $(F \otimes_{\mathbb{Z}} \mathbb{Q})/K$ is divisible, being the quotient of a divisible abelian group (see the above exercise), we are done in the case $X = \{x\}$.

In the general case, for each $x \in X$, choose an injective abelian group I_x and an embedding $\mathcal{F}_x \hookrightarrow I_x$. For each $x \in X$, let $\varphi_x \colon \{x\} \hookrightarrow X$ denote the natural inclusion. We define

$$\mathcal{I} := \prod_{x \in X} (\varphi_x)_*(I_x).$$

We have

$$\operatorname{Hom}(\mathcal{F},\mathcal{I}) = \prod_{x\in X}(\mathcal{F},(\varphi_x)_* I_x) = \prod_{x\in X}\operatorname{Hom}(\mathcal{F}_x, I_x).$$

This yields a natural morphism of sheaves $\mathcal{F}\to\mathcal{I}$, which is injective since it is so on each stalk. It is also easily checked that $\mathcal{I}$ is injective. We are done. □

Definition 2.2.12. Let $\mathcal{F}$ be an abelian sheaf on a topological space X. An *injective resolution* of $\mathcal{F}$ is a complex $\mathcal{I}^\bullet$, defined in degrees $i\geq 0$, together with a morphism $\epsilon\colon \mathcal{F}\to\mathcal{I}^0$ such that $\mathcal{I}^i$ is injective for each $i\geq 0$ and such that the sequence

$$0\to\mathcal{F}\to\mathcal{I}^0\to\mathcal{I}^1\to\cdots$$

is exact.

Corollary 2.2.13. *Let X be a topological space. Then any abelian sheaf $\mathcal{F}$ on X admits an injective resolution.* □

Lemma 2.2.14. *Let X be a topological space and let $\mathcal{F}\to\mathcal{I}^\bullet$ and $\mathcal{F}\to\mathcal{J}^\bullet$ be two injective resolutions. Then there are morphisms of complexes $f\colon\mathcal{I}^\bullet\to\mathcal{J}^\bullet$ and $g\colon\mathcal{J}^\bullet\to\mathcal{I}^\bullet$ whose compositions are homotopic to the identity (see Definition 2.1.3).*

Proof. Exercise. □

Note that if $\mathcal{I}^\bullet$ is an injective resolution of an abelian sheaf $\mathcal{F}$ on X, we get a complex $\Gamma(X,\mathcal{I}^\bullet)$ whose terms are $\Gamma(X,\mathcal{I}^i)=\mathcal{I}^i(X)$ for $i\geq 0$.

Definition 2.2.15. Let X be a topological space. For each abelian sheaf $\mathcal{F}$ on X, choose an injective resolution $\mathcal{F}\to\mathcal{I}^\bullet$, and define $\mathrm{H}^i(X,\mathcal{F})=\mathrm{H}^i(\Gamma(X,\mathcal{I}^\bullet))$.

Theorem 2.2.16. *Let X be a topological space.*

(1) For each $i\geq 0$, the assocation $\mathcal{F}\mapsto\mathrm{H}^i(X,\mathcal{F})$ defines a functor from $\mathsf{AbSh}(X)$ to Ab. Moreover, this functor is, up to natural isomorphism of functors, independent of the choices of injective resolutions made.

(2) We have $\mathrm{H}^0(X,\mathcal{F})=\mathcal{F}(X)$.

(3) Let $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3$ be a short exact sequence of abelian sheaves. Then there is an associated long exact sequence in cohomology:

$$\cdots\to\mathrm{H}^i(X,\mathcal{F}_1)\to\mathrm{H}^i(X,\mathcal{F}_2)\to\mathrm{H}^i(X,\mathcal{F}_3)\to\mathrm{H}^{i+1}(X,\mathcal{F}_1)\to\mathrm{H}^{i+1}(X,\mathcal{F}_2)\to\cdots.$$

Proof. Exercise. *Hint:* Use Lemmas 2.2.14 and 2.1.2. □

Theorem 2.2.17. *Let X be a noetherian separated scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a canonical isomorphism between the group $\mathrm{H}^p(X,\mathcal{F})$ introduced in Definition 2.2.15 and the Cech cohomology group $\mathrm{H}^p(\mathcal{U},\mathcal{F})$ introduced in Definition 2.1.8, where $\mathcal{U}=\{U_0,\ldots,U_r\}$ is a finite cover of affine opens $U_i\subset X$.*

Proof. Exercise. □

2.3 Lecture 18 : Coherent sheaves on projective schemes

2.3.1 Cohomology of twisting sheaves on projective space

Recall. *See Examples 2.2.2, 2.2.3 and 2.2.4. We have* $\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}(m)) = k[x_0, x_1]_m$, $\mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}(m)) = 0$ *for* $m \geq -1$, *and* $\dim \mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}(m)) = -m-1$ *for* $m \geq -2$.

We would like to generalize this to projective spaces of arbitrary dimension $n \geq 1$.

Theorem 2.3.1. *Let* $\mathbb{P}^n_A = \mathrm{Proj} A[x_0, \ldots, x_n]$ *where* A *is a noetherian ring. Then:*

(1) For each $m \in \mathbb{Z}$, $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(m)) = A[x_0, \ldots, x_n]_m$.

(2) For all $0 < p < n$ *and all* $m \in \mathbb{Z}$, $\mathrm{H}^p(\mathbb{P}^n_A, \mathcal{O}(m)) = 0$.

(3) For each $m \in \mathbb{Z}$,

$$\mathrm{H}^n(\mathbb{P}^n_A, \mathcal{O}(m)) = \left(x_0^{-1} \cdots x_n^{-1} \cdot A[x_0^{-1}, \ldots, x_n^{-1}]\right)_m .$$

In particular, $\mathrm{H}^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) = A$.

Proof. We consider the open cover $\mathcal{U} = \{U_i\}$ with $U_i = D_+(x_i)$. This gives

$$I = \{0, \ldots, n\} .$$

We get

$$\mathcal{C}^p(\mathcal{U}, \mathcal{O}(m)) = \prod_{i_0 < \cdots < i_p} \Big(A[x_0, \ldots, x_n]_{x_{i_0} \cdots x_{i_p}}\Big)_m .$$

The Cech complex takes the form

$$\prod_i (A[x_0, \ldots, x_n]_{x_i})_m \xrightarrow{d_0} \prod_{i<j} \big(A[x_0, \ldots x_n]_{x_i x_j}\big)_m \xrightarrow{d_1} \prod_{i<j<k} \big(A[x_0, \ldots, x_n]_{x_i x_j x_k}\big)_m \xrightarrow{d_2} \cdots .$$

For each $i_0 < \cdots < i_p \in I$, we have a decomposition

$$\Big(A[x_0, \ldots, x_n]_{x_{i_0} \cdots x_{i_p}}\Big)_m = \bigoplus_{\substack{e \in \mathbb{Z}^{n+1} :\ \deg(e) = m \\ e_j \geq 0\ \forall j \notin \{i_0, \ldots, i_p\}}} A x_0^{e_0} \cdots x_n^{e_n} .$$

This gives a decomposition

$$\mathcal{C}^p(\mathcal{U}, \mathcal{O}(m)) = \prod_{i_0 < \cdots < i_p} \Big(A[x_0, \ldots, x_n]_{x_{i_0} \cdots x_{i_p}}\Big)_m = \prod_{i_0 < \cdots < i_p} \bigoplus_{\substack{e \in \mathbb{Z}^{n+1} :\ \deg(e) = m \\ e_j \geq 0\ \forall j \notin \{i_0, \ldots, i_p\}}} A x_0^{e_0} \cdots x_n^{e_n} .$$

Note that (1) follows from Proposition 1.2.3. Let us prove (2) and (3). We have:

$$(A[x_0, \ldots, x_n]_{x_0 \cdots x_n})_m = \bigoplus_{\sum e_i = m} A x_0^{e_0} \cdots x_n^{e_n} .$$

More generally:

$$C^p(\mathcal{U}, \mathcal{O}(m)) = \bigoplus_{e \in \mathbb{Z}^{n+1}} C^p(\mathcal{U}, \mathcal{O}(m))_e,$$

with

$$\mathcal{C}^p(\mathcal{U}, \mathcal{O}(m))_e = \prod_{i_0 < \cdots < i_p :\, e_j \geq 0 \ \forall j \notin \{i_0, \ldots, i_p\}} (x_0^{e_0} \cdots x_n^{e_n} A)_m .$$

Therefore, to prove (ii), it suffices to prove that the complex $C^\bullet(\mathcal{U}, \mathcal{O}(m))_e$ is exact in the range $0 < p < n$, for each $e \in \mathbb{Z}^{n+1}$. For $\deg(e) \neq m$, the complex is zero. For $\deg(e) = m$ and $0 \leq p \leq n$, we have a canonical split embedding

$$\prod_{\substack{i_0 < \cdots < i_p \leq n \\ e_j \geq 0 \ \forall j \notin \{i_0, \ldots, i_p\}}} x_0^{e_0} \cdots x_n^{e_n} A \hookrightarrow \prod_{i_0 < \cdots < i_p \leq n} x_0^{e_0} \cdots x_n^{e_n} A,$$

and the complex

$$\to \prod_{i_0 < \cdots < i_{p-1} \leq n} x_0^{e_0} \cdots x_n^{e_n} A \to \prod_{i_0 < \cdots < i_p \leq n} x_0^{e_0} \cdots x_n^{e_n} A \to \prod_{i_0 < \cdots < i_{p+1} \leq n} x_0^{e_0} \cdots x_n^{e_n} \cdot A \to \cdots$$

identifies with the complex $C^\bullet$ with $C^p = \prod_{i_0 < \cdots < i_p} A$, that is, with

$$\to \prod_{i_0 < \cdots < i_{p-1} \leq n} A \to \prod_{i_0 < \cdots < i_p \leq n} A \to \cdots \to \prod_{i_0 < i_1 < \cdots < i_n} A = A.$$

The latter is exact in degrees $0 < p < n$ (see Example 2.2.8), hence the former is exact in those degrees as well. This proves (2).

To prove (3), observe that

$$\mathcal{C}^n(\mathcal{U}, \mathcal{O}(m)) = (A[x_0, \ldots, x_n]_{x_0 \cdots x_n})_m$$

is a free graded A-module spanned by the monomials of the form $x_0^{e_0} \cdots x_n^{e_n}$ with $\sum e_i = m$. The image of d^{n-1} is spanned by the monomials $x_0^{e_0} \cdots x_n^{e_n}$ with $\sum e_i = m$ and at least one $e_j \geq 0$. Hence

$$\begin{aligned} \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(m)) = \mathrm{Coker}(d^{n-1}) &= A \left\{ x_0^{e_0} \cdots x_n^{e_n} \mid e_i < 0 \ \forall i \text{ and } \sum e_i = m \right\} \\ &= \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}] \right)_m . \end{aligned}$$

This gives

$$\mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(-n-1)) = \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}] \right)_{-n-1} = A \cdot x_0^{-1} \cdots x_n^{-1}.$$

The proof is finished. □

Corollary 2.3.2. *Let k be a field. For $m \geq 0$, we have*

$$\dim \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(m)) = \binom{m+n}{n}. \tag{2.1}$$

Moreover, for $m \leq -n-1$, we have

$$\dim \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(m)) = \binom{-m-1}{n}.$$

We have $\mathrm{H}^p(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for all other values of $(p, m) \in \mathbb{Z}^{\oplus 2}$.

Proof. Remark that (2.1) holds by item (1) of Theorem 2.3.1. Indeed, we have that $\dim k[x_0, \ldots, x_n]_m = \binom{m+n}{n}$. By item (3) of Theorem 2.3.1, we have $\mathrm{H}^n(\mathbb{P}^n_k, \mathcal{O}(m)) = (x_0 \cdots x_n)^{-1} \cdot k[x_0^{-1}, \ldots, x_n^{-1}]$. Now note that there are natural isomorphisms

$$\left((x_0 \cdots x_n)^{-1} \cdot k[x_0^{-1}, \ldots, x_n^{-1}]\right)_m = k[x_0^{-1}, \ldots, x_n^{-1}]_{m+n+1} = k[t_0, \ldots, t_n]_{-m-n-1}.$$

Therefore,

$$\dim \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(m)) = \dim\left(k[t_0, \ldots, t_n]_{-m-n-1}\right) = \binom{(-m-n-1)+n}{n} = \binom{-m-1}{n}.$$

The corollary follows. $\square$

2.3.2 Cohomology of coherent sheaves on projective schemes

Theorem 2.3.3. *Let A be a noetherian ring. Let $X \subset \mathbb{P}^r_A$ be a projective scheme over A. For $n \in \mathbb{Z}$, consider the sheaf $\mathcal{O}_X(n)$ on X. Let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(1) The cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are finitely generated A-modules for each $i \geq 0$.

(2) There exists an $n_0 > 0$ such that

$$\mathrm{H}^i(X, \mathcal{F}(n)) = 0 \qquad (\textit{where } \mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))$$

for all $n \geq n_0$ and $i > 0$.

Example 2.3.4. Let X be an integral projective scheme over a field k. Then

$$\dim_k \mathrm{H}^0(X, \mathcal{O}_X) = 1.$$

Indeed, consider a closed immersion $i \colon X \hookrightarrow \mathbb{P}^n_k$ for some $n \geq 0$. Then $X = \mathrm{Proj}(k[x_0, \ldots, x_n]/I)$ for some homogeneous ideal $I \subset k[x_0, \ldots, x_n]$, see Proposition 1.2.13. Now $S \coloneqq k[x_0, \ldots, x_n]/I$ is a graded k-algebra which is a domain (since X is integral), generated by elements $x_0, \ldots, x_n \in S_1$ over k which are relatively prime. Thus, by Exercise 1.2.4, the map $\beta \colon S \to \Gamma_*(\mathcal{O}_X)$ defined in (1.4) is an isomorphism. In particular,

$$k = S_0 = (\Gamma_*(\mathcal{O}_X))_0 = \Gamma(X, \mathcal{O}_X) = \mathrm{H}^0(X, \mathcal{O}_X).$$

To prove Theorem 2.3.3, we need a couple of results.

Lemma 2.3.5. *Let X be a topological space and let $i\colon Z \subset X$ be a closed subset. Let $\mathcal{U}$ be an open cover of X, and let $\mathcal{U}_Z$ be the induced open cover of Z. Then for any sheaf $\mathcal{F}$ on Z and any $p \geq 0$, we have $\mathrm{H}^p(Z, \mathcal{F}) = \mathrm{H}^p(X, i_*\mathcal{F})$.*

Proof. This follows from the fact that for each open $U \subset X$, $\Gamma(U \cap Z, \mathcal{F}) = \Gamma(U, i_*\mathcal{F})$, so the two cohomolgy groups arise from the same Cech complexes. $\square$

Lemma 2.3.6. *Let $f\colon X \to Y$ be a morphism of schemes. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{L}$ be a line bundle on Y. Then there exists an isomorphism*

$$\varphi\colon f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L} \xrightarrow{\sim} f_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})\right). \tag{2.2}$$

Proof. Let $\{U_i\}$ be an open cover of Y such that for each $i \in I$ there exists an isomorphism $\rho_i\colon \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. For $i \in I$, define an isomorphism

$$\varphi_i\colon \left(f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L}\right)|_{U_i} \xrightarrow{\sim} \left(f_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})\right)\right)|_{U_i}$$

as the composition

$$\left(f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L}\right)|_{U_i} \cong f_*(\mathcal{F})|_{U_i} \cong \left(f_*\left(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})\right)\right)|_{U_i}.$$

Note that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. Thus, the φ_i glue to an isomorphism (2.2). $\square$

Lemma 2.3.7. *Let S be a graded ring and let M be a finitely generated graded S-module. Then M is generated by finitely many homogeneous elements, and there is a set of integers $a_1, \ldots, a_n \in \mathbb{Z}$ and a surjection of graded S-modules $\oplus_i S(-a_i) \to M$.*

Proof. First observe that there exists a set of generators $\{m_1, \ldots, m_n\} \subset M$ for M over S such that each m_i is homogeneous. Let $a_i = \deg(m_i)$. The map $S(-a_i) \to M$ that sends $1 \in S(a_i)_{a_i} = S_0$ to the element m_i is a morphism of graded S-modules. Moreover, the resulting map of graded S-modules $\oplus_i S(-a_i) \to M$ is surjective. $\square$

Proof of Theorem 2.3.3. Let $i\colon X \hookrightarrow \mathbb{P}^r_A$ be the given closed embedding into $\mathbb{P}^r_A$. Then $i_*\mathcal{F}$ is coherent and

$$\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^i(\mathbb{P}^r_A, i_*\mathcal{F}),$$

see Lemma 2.3.5. Moreover, by Lemma 2.3.6, we have $\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n) = i_*\left(\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n)\right)$, so that

$$\begin{aligned}
\mathrm{H}^i(X, \mathcal{F}(n)) &= \mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{O}_X(n)) \\
&= \mathrm{H}^i(X, \mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n)) \\
&= \mathrm{H}^i(\mathbb{P}^r, i_*\left(\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^r_A}(n)\right)) \\
&= \mathrm{H}^i(\mathbb{P}^r_A, i_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r_A}(n)).
\end{aligned}$$

This reduces the theorem to the case $X = \mathbb{P}^r_A$.

Recall (see Proposition 1.2.8) that in this case, the coherent sheaf $\mathcal{F}$ on $X = \mathbb{P}^r_A$ is of the form $\mathcal{F} = \widetilde{M}$ for some finitely generated graded S-module M, where $S = A[x_0, \dots, x_n]$. Both parts of the theorem are trivially satisfied when $i > \dim \mathbb{P}^r_A = r + \dim(A)$. We take this as the base case, and proceed by downwards induction on i.

(1). As M is finitely generated, we may pick a surjection of graded A-modules

$$\bigoplus_k A(-a_k) \longrightarrow M.$$

The kernel K of this surjection is graded and finitely generated (see Lemma 1.1.4), so that we get an exact sequence of finitely generated graded A-modules

$$0 \to K \to \bigoplus_k A(-a_k) \to M \to 0.$$

Applying the tilde functor, which is exact by Lemma 1.1.11, we get an exact sequence of coherent sheaves

$$0 \to \mathcal{K} = \widetilde{K} \to \bigoplus_k \mathcal{O}_{\mathbb{P}^r_A}(-a_k) \to \mathcal{F} \to 0. \tag{2.3}$$

Taking the long exact sequence in cohomology yields:

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{K}) \to \bigoplus_k \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^r_A}(-a_k)) \to \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{F}) \to \mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K}) \to \cdots.$$

By the induction hypothesis, we have that $\mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K})$ is a finitely generated A-module. The A-module $\bigoplus_k \mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^r_A}(-a_k))$ is also finitely generated, see Theorem 2.3.1. Hence, we get that $\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated.

(2). It suffices to prove that for each $i > 0$, there exists $n_0 > 0$ such that $\mathrm{H}^i(\mathbb{P}^r_A, \mathcal{F}(n)) = 0$ for all $n \geq n_0$. Indeed, one then takes the max of all such n_0 defined for the various $0 < i \leq r + \dim(A)$.

Twist the exact sequence (2.3) by $\mathcal{O}_{\mathbb{P}^r_A}(n)$ and take cohomology, to get an exact sequence

$$\cdots \to \mathrm{H}^i(\mathbb{P}^r_A, \mathcal{K}(n)) \to \bigoplus_k \mathrm{H}^i(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n-a_k)) \to \mathrm{H}^i(\mathbb{P}^r_A, \mathcal{F}(n)) \to \mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K}(n)) \to \cdots.$$

Again, by downward induction on $i > 0$, we get some n_0 such that $\mathrm{H}^{i+1}(\mathbb{P}^r_A, \mathcal{K}(n)) = 0$ for $n \geq n_0$, and enlarging n_0 if necessary, we may assume $\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}(n - a_k)) = 0$ for $n \geq n_0$ and all k (see Theorem 2.3.1. This gives $\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{F}(n)) = 0$ for $n \geq n_0$. $\square$

2.3.3 *Picard group of a scheme*

Definition 2.3.8. Let X be a scheme.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *finite locally free* if there exists an open covering $\{U_i\}_{i \in I}$ of X together with an integer $n_i \in \mathbb{Z}_{\geq 1}$ and an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{n_i}$ for each $i \in I$.

(2) An $\mathcal{O}_X$-module $\mathcal{L}$ is *invertible* if there exists an open covering $\{U_i\}$ of X and for each i an isomorphism $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ of $\mathcal{O}_{U_i}$-modules. We call $\mathcal{L}$ a *line bundle*.

(3) We let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of line bundles on X.

Exercise 2.3.9. Let X be a scheme. For a line bundle $\mathcal{L}$ on X, show that the $\mathcal{O}_X$-module

$$\mathcal{L}^{-1} := \mathscr{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$$

is a line bundle on X. Show that $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ is a line bundle on X if $\mathcal{L}_1, \mathcal{L}_2$ are line bundles on X. Show that $\mathrm{Pic}(X)$ admits a natural structure of an abelian group. It is called the *Picard group* of X.

Exercise 2.3.10. Let k be a field. Show that $\mathrm{Pic}(\mathbb{A}^1_k) = 0$.

Exercise 2.3.11. Let k be a field. Let $X = \mathbb{P}^1_k$.

(1) Show that $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.

(2) Show that

$$\mathrm{Pic}(\mathbb{P}^1_k) = \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^1_k}(1).$$

In other words, for any line bundle $\mathcal{L}$ on $X = \mathbb{P}^1_k$, we have $\mathcal{L} \cong \mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$.

2.4 Lecture 19 : Hypersurfaces

2.4.1 Field-valued points of schemes

Let k be a field and let X be a scheme over k.

Definition 2.4.1. For a scheme T over k, we write

$$X(T) := \mathrm{Hom}_{\mathsf{Sch}/k}(T, X).$$

This is the set of morphisms of k-schemes $T \to X$. If $T = \mathrm{Spec}\ A$ is affine, we write $X(A) = X(T)$.

Note that for affine k-schemes $X = \mathrm{Spec}\ R$ and $T = \mathrm{Spec}\ A$, we have that $X(T) = X(A)$ is naturally in bijection with the set of morphisms of k-algebras $R \to A$.

Lemma 2.4.2. *Suppose that $X = \mathrm{Spec}\ R$ with*

$$R = k[t_1, \dots, t_n]/(f_1, \dots, f_m), \qquad f_i \in k[t_1, \dots, t_n].$$

Let $T = \mathrm{Spec}\ A$ be an affine scheme over k. Then there are natural bijections

$$\begin{aligned} X(A) = X(T) &= \mathrm{Hom}_{\mathsf{Sch}/k}(T, X) \\ &= \mathrm{Hom}_{k\text{-}\mathsf{Alg}}(R, A) = \{\alpha \in A^n \mid f_i(\alpha) = 0\ \forall i \in \{1, \dots, m\}\}. \end{aligned}$$

Proof. Exercise. $\square$

Examples 2.4.3. (1) Let $X = \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2)$. Then $X(\mathbb{R}) = \emptyset$.

(2) Let $X = \operatorname{Spec} \mathbb{R}[x,y]/(x+y, x-y)$. Then $X(\mathbb{R}) = \{(0,0)\} \subset \mathbb{A}^2(\mathbb{R}) = \mathbb{R}^2$.

Example 2.4.4. Let k be a field. Let $V = k^{n+1}$. Then there is a natural isomorphism of k-vector spaces $V \xrightarrow{\sim} V^\vee$ given by $e_i \mapsto e_i^\vee$. This gives an isomorphism

$$\mathbb{P}_k^n = \check{\mathbb{P}}_k^r,$$

where we recall that

$$\check{\mathbb{P}}_k^r = \mathbb{P}(V^\vee) \quad \text{and that} \quad \mathbb{P}(W) = \operatorname{Proj}(\operatorname{Sym}^*(W))$$

for a finite dimensional k-vector space W. For each field extension $k' \supset k$, one gets a canonical bijection (see also Example 1.2.20):

$$\mathbb{P}_k^n(k') = \left\{\text{lines } \ell \subset (k')^{n+1}\right\}.$$

2.4.2 Hypersurfaces in projective space

Definition 2.4.5. (1) A *hypersurface* is a closed subscheme $X \subset \mathbb{P}_k^n$ defined as

$$X = V(F) = \operatorname{Proj}(k[x_0, \ldots, x_n]/(F)),$$

for some homogeneous polynomial $F \in k[x_0, \ldots, x_n]$ of positive degree. The *degree* of this hypersurface is the degree of F.

(2) A *complete intersection of two hypersurfaces* $X \subset \mathbb{P}_k^n$ is a closed subscheme

$$X = V(F) \cap V(G) = V(F,G) \subset \mathbb{P}_k^n$$

defined by two homogeneous polynomials $F, G \in k[x_0, \ldots, x_n]$ of positive degrees $d > 0, e > 0$ such that $V(F)$ and $V(G)$ have no irreducible component in common.

Example 2.4.6. Continue with the notation from Example 2.4.4. Let $X = V(F) \subset \mathbb{P}_k^n$ be a hypersurface. Then for each field extension $k' \supset k$, we have:

$$X(k') = \{\alpha = [x_0 \colon \cdots \colon x_n] \in \mathbb{P}^n(k') \mid F(\alpha) = 0\} \subset \mathbb{P}^n(k').$$

Exercise 2.4.7. For a hypersurface $X = V(F) \subset \mathbb{P}_k^n$ of degree $d > 0$, show that:

(1) $\dim(X) = n - 1$;

(2) the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}_k^n}$ is canonically isomorphic to the sheaf $\mathcal{O}_{\mathbb{P}_k^n}(-d)$.

Exercise 2.4.8. For a complete intersection $X = V(F) \cap V(G) = V(F,G) \subset \mathbb{P}_k^n$, where $\deg(F) = d > 0$ and $\deg(G) = e > 0$, show that:

(1) $\dim(X) = n - 2$:

(2) for $R = k[x_0, \ldots, x_n]$, the sequence of graded R-modules

$$0 \to R(-d-e) \xrightarrow{\alpha} R(-d) \oplus R(-e) \xrightarrow{\beta} (F, G) \to 0$$

is exact, where $\alpha(h) = (-hG, hF)$ and $\beta(h_1, h_2) = h_1 F + h_2 G$.

(3) Applying the tilde functor, we get an exact sequence of $\mathcal{O}_{\mathbb{P}^n_k}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_X \to 0,$$

where $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n_k}$ is the ideal sheaf of $X \subset \mathbb{P}^n_k$.

2.4.3 Genus of a plane curve

Definition 2.4.9. Let k be a field.

(1) A *curve* over k is an integral scheme C which is separated and of finite type over k, with $\dim(C) = 1$.

(2) The *genus* $g(C)$ of a projective curve C is the dimension of the k-vector space $\mathrm{H}^1(C, \mathcal{O}_C)$. This dimension is finite by Theorem 2.3.3.

(3) A *plane curve* is a hypersurface $C \subset \mathbb{P}^2_k$ which is integral. Remark that any plane curve is a curve.

Example 2.4.10. The projective line $\mathbb{P}^1_k$ is a curve with $g(\mathbb{P}^1_k) = 0$.

Definition 2.4.11. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a plane curve defined by a homogeneous polynomial $F \in k[x_0, x_1, x_2]$ of positive degree. We say that C is *smooth* if there is no point $p \in C(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ such that $\partial F / \partial x_i(p) = 0$ for each $i = 0, 1, 2$. In other words, C is smooth if there is no $p \in \mathbb{P}^2(\mathbb{C})$ such that

$$F(p) = \partial F/\partial x_0(p) = \partial F/\partial x_1(p) = \partial F/\partial x_2(p) = 0.$$

Proposition 2.4.12. *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth plane curve. Then, with respect to the natural complex manifold structure of $\mathbb{P}^2(\mathbb{C})$, we have that $C(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ is a complex submanifold of dimension one.*

In particular, $C(\mathbb{C})$ is a connected and compact Riemann surface in a natural way.

Proof. Exercise. □

Fact 2.4.13. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth plane curve. Then $g(C)$ equals the (topological) genus of the Riemann surface $C(\mathbb{C})$. In particular, $\mathrm{rank}_{\mathbb{Z}} H^1(C(\mathbb{C}), \mathbb{Z}) = 2 \cdot g(C)$.

Lemma 2.4.14. *Let $n \in \mathbb{Z}_{\geq 3}$ and let $0 \to V_1 \to \cdots \to V_n \to 0$ be an exact complex of finite dimensional vector spaces V^i over a field k. Then $\sum_{i=1}^n (-1)^i \dim(V_i) = 0$.*

Proof. First assume $n = 3$. If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of finite dimensional vector spaces, then there exists a injective linear map $V_3 \to V_2$ whose composition with the given map $V_2 \to V_3$ is the identity: the sequence *splits*. Thus $V_2 \cong V_1 \oplus V_3$ in this case, whence the result.

We assume $n \geq 4$ and apply induction on n, assuming the lemma to be true for $n-1$. Let $W_{n-1} = \operatorname{Coker}(V_{n-3} \to V_{n-2})$. Then we have exact sequences $0 \to V_1 \to \cdots \to V_{n-3} \to V_{n-2} \to W_{n-1} \to 0$ and $0 \to W_{n-1} \to V_{n-1} \to V_n \to 0$. By the induction hypothesis, we have

$$\sum_{i=1}^{n-2}(-1)^i \dim(V_i) + (-1)^{n-1}\dim(W_{n-1}) = 0.$$

Moreover, the $n = 3$ case gives $(-1)^{n-1}\dim(W_{n-1}) = (-1)^{n-1}(\dim(V_{n-1}) - \dim(V_n))$. Hence,

$$\begin{aligned}
0 &= \sum_{i=1}^{n-2}(-1)^i \dim(V_i) + (-1)^{n-1}\dim(W_{n-1}) \\
&= \sum_{i=1}^{n-2}(-1)^i \dim(V_i) + (-1)^{n-1}(\dim(V_{n-1}) - \dim(V_n)) \\
&= \sum_{i=1}^{n-1}(-1)^i \dim(V_i) + (-1)^{n}\dim(V_n) \\
&= \sum_{i=1}^{n}(-1)^i \dim(V_i).
\end{aligned}$$

We are done. □

Theorem 2.4.15. *Let $C \subset \mathbb{P}^2_k$ be a plane curve of degree $d > 0$. Then*

$$g(C) = (d-1)(d-2)/2.$$

Proof. Let $i\colon C \hookrightarrow \mathbb{P}^2_k$ be the natural closed immersion. Consider the ideal sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_C \to 0.$$

Using Lemma 2.3.5, we get a long exact sequence

$$\begin{aligned}
0 &\to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(-d)) \to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to \mathrm{H}^0(C, \mathcal{O}_C) \to \\
&\to \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}(-d)) \to \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to \mathrm{H}^1(C, \mathcal{O}_C) \to \\
&\to \mathrm{H}^2(\mathbb{P}^2, \mathcal{O}(-d)) \to \mathrm{H}^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to 0.
\end{aligned}$$

In view of Lemma 2.4.14 and Corollary 2.3.2, this gives:

$$0 - 1 + 1 - 0 + 0 - g(X) + \binom{d-1}{2} - 0 = 0.$$

Therefore,

$$g(X) = \binom{d-1}{2} = \frac{(d-1)!}{2!(d-3)!} = \frac{(d-1)(d-2)}{2}.$$

This proves the proposition. $\square$

Example 2.4.16. Consider $\mathbb{P}^2_{\mathbb{C}} = \mathrm{Proj}\mathbb{C}[x, y, z]$. Let

$$C = V(zy^2 - x^3 - z^3) \subset \mathbb{P}^2_{\mathbb{C}}.$$

Then C is smooth (see Definition 2.4.11), and the Riemann surface $C(\mathbb{C})$ is topologically a torus. Hence $g(C) = 1$ (see Fact 2.4.13). This is compatible with Theorem 2.4.15, since $1 = (3-1)(3-2)/2$.

Chapter 3

Divisors

3.1 Lecture 20 : Bézout's theorem and Weil divisors

3.1.1 Bézout's theorem

Let k be an algebraically closed field. Let $C \subset \mathbb{P}^2_k$ and $D \subset \mathbb{P}^2_k$ be two plane curves of degrees $d > 0$ and $e > 0$, that have no irreducible component in common. This implies that the scheme-theoretic intersection

$$Z = C \times_{\mathbb{P}^2_k} D \subset \mathbb{P}^2_k$$

is a zero-dimensional subscheme of $\mathbb{P}^2_k$. In particular, the underlying topological space $|Z|$ of Z consists of finitely many closed points $p_1, \ldots, p_r \in |\mathbb{P}^2_k|$. Note that there exists an automorphism $\phi \in \mathrm{Aut}(\mathbb{P}^2_k)$ such that $\phi(|Z|)$ is contained in the affine open

$$U_0 \coloneqq D_+(x_0) = \mathrm{Spec}\ (k[x_0, x_1, x_2]_{(x_0)}) \cong \mathrm{Spec}\ (k[x, y]).$$

Replacing C by $\phi(C)$ and D by $\phi(D)$, we get that $Z \subset U_0 \subset \mathbb{P}^2_k$. Let

$$\mathfrak{m}_i \subset k[x, y]$$

be the maximal ideal associated to the closed point $p_i \in U_0 = \mathrm{Spec}\ k[x, y] = \mathbb{A}^2_k$.

Theorem 3.1.1 (Bézout's theorem)**.** *Under the above notation and assumptions,*

$$\dim \mathrm{H}^0(Z, \mathcal{O}_Z) = \sum_{i=1}^{r} \dim_k \left(\frac{k[x, y]}{(f, g)} \right)_{\mathfrak{m}_{z_i}} = d \cdot e.$$

Example 3.1.2. Let $C = V(x_1 - x_2)$ and $D = V(x_1 + x_2)$. Then $Z = C \times_{\mathbb{P}^2_k} D = V(x_1 - x_2, x_1 + x_2) = V(x_1, x_2) \subset U_0$. We get $Z = \mathrm{Spec}\ k$ with closed embedding $\mathrm{Spec}\ k \hookrightarrow U_0 = \mathbb{A}^2_k$ given by $0 \in \mathbb{A}^2_k(k) = \mathrm{Hom}_{\mathsf{Sch}/k}(\mathrm{Spec}\ k, \mathbb{A}^2_k)$, see Lemma 2.4.2.

Proof of Theorem 3.1.1. Since Z is a zero-dimensional subscheme of $U_0 = \mathrm{Spec}\ k[x, y]$, it is clear that

$$\mathcal{O}_Z(Z) = \bigoplus_{i=1}^{r} \mathcal{O}_{Z, p_i},$$

and that

$$\mathcal{O}_{Z,p_i} = \mathcal{O}_{U_0,p_i}/\mathcal{I}_{Z,p_i} = (\mathcal{O}(U_0)/\mathcal{I}_Z(U_0))_{\mathfrak{m}_i} = \left(\frac{k[x,y]}{(f,g)}\right)_{\mathfrak{m}_{z_i}} \qquad \forall i \in \{1,\ldots,r\}\,.$$

Moreover, for the natural closed immersion $i\colon Z \hookrightarrow \mathbb{P}^2_k$, we have the ideal sheaf sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^2_k} \to i_*\mathcal{O}_Z \to 0$, which gives exact sequences

$$0 \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \to \mathrm{H}^0(Z, \mathcal{O}_Z) \to \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) \to 0$$

and

$$0 = \mathrm{H}^1(Z, \mathcal{O}_Z) = \mathrm{H}^1(\mathbb{P}^2, i_*\mathcal{O}_Z) \to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0,$$

where $\mathrm{H}^1(Z, \mathcal{O}_Z) = 0$ because $\dim(Z) = 0$. This gives:

$$\begin{aligned} \dim \mathrm{H}^0(Z, \mathcal{O}_Z) &= \dim \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) - \dim \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z), \\ \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) &= 0. \end{aligned}$$

Recall the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^2_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^2_k}(-e) \to \mathcal{I}_Z \to 0, \tag{3.1}$$

see Exercise 2.4.8. As $\mathrm{H}^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(m)) = 0$ for each $m \in \mathbb{Z}$, see Corollary 2.3.2, we get an exact sequence

$$0 = \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^2_k}(-e)) \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d-e)) = 0,$$

which shows that $\mathrm{H}^0(\mathbb{P}^2_k, \mathcal{I}_Z) = 0$. Hence

$$\dim \mathrm{H}^0(Z, \mathcal{O}_Z) = \dim \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) = 1 + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z).$$

Furthermore, (3.1) gives a long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) \to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d-e)) &\to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \oplus \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-e)) \\ &\to \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) = 0, \end{aligned}$$

where the vanishing $\mathrm{H}^2(\mathbb{P}^2_k, \mathcal{I}_Z) = 0$ has been shown above. We conclude that

$$\begin{aligned} \dim_k \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) &= \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d-e)) \\ &\quad - \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) - \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-e)) \\ &= \binom{d+e-1}{2} - \binom{d-1}{2} - \binom{e-1}{2}, \end{aligned}$$

see Corollary 2.3.2. Now

$$\begin{aligned} &\binom{d+e-1}{2} - \binom{d-1}{2} - \binom{e-1}{2} \\ &\qquad = \frac{(d+e-1)(d+e-2)}{2} - \frac{(d-1)(d-2)}{2} - \frac{(e-1)(e-2)}{2} \\ &\qquad = \frac{1}{2} \cdot \big((d^2 + 2de - 3d + e^2 + 2) - (d^2 - 3d + 2) - (e^2 - 3e + 2)\big) \\ &\qquad = \frac{2de - 2}{2} = de - 1. \end{aligned}$$

Therefore,

$$\dim_k \mathrm{H}^0(Z, \mathcal{O}_Z) = 1 + \dim \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{I}_Z) = 1 + de - 1 = de.$$

The theorem follows. $\square$

3.1.2 Definition of an algebraic variety

In this course, we follow the Stacks Project with our notion of *algebraic variety*:

Definition 3.1.3. Let k be a field.

(1) An *algebraic variety* (or simply a *variety*) over k is a scheme X over k such that X is integral, and such that the structure morphism $X \to \operatorname{Spec} k$ is separated and of finite type.

(2) A *curve* (resp. *surface*, resp. *threefold*) is an algebraic variety of dimension one (resp. two, resp. three).

Remark 3.1.4. Suppose that k'/k is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_k k'$ is not necessarily a variety over k'. For instance, let $k = \mathbb{Q}$, let $X = \operatorname{Spec} \mathbb{Q}(i)$ and let $k' = \operatorname{Spec} \mathbb{Q}(i)$. Then

$$X_{k'} = \operatorname{Spec} \left(\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i)\right) \cong \operatorname{Spec} \mathbb{Q}(i) \sqcup \operatorname{Spec} \mathbb{Q}(i).$$

Remark 3.1.5. The same counterexample shows that the product of two varieties need not be a variety. If the ground field is algebraically closed however, then the product of varieties X and Y over $k = \bar{k}$ is a variety over k. This statement readily reduces to the affine case, and in fact to the statement that for an algebraically closed field k and two finitely generated k-algebras A and B which are integral domains, the tensor product $A \otimes_k B$ is an integral domain. We leave this as an exercise.

Corollary 3.1.6. *Let*

$$X \to \operatorname{Spec} k$$

be a projective morphism, where k is a field and X is a scheme. Then X is separated and of finite type over k. In particular, if X is integral, then X is a variety over k.

Proof. Indeed, the composition of two separated (resp. finite type) morphisms is separated (resp. of finite type), and $\mathbb{P}^n_k$ is separated and of finite type over k. $\square$

Example 3.1.7. Let C be a curve over a field k. Then C is an algebraic variety.

Example 3.1.8. Let $X = \operatorname{Spec} \mathbb{C}$ and consider the morphism $X \to \operatorname{Spec} \mathbb{R}$. This turns X into an algebraic variety over $\mathbb{R}$.

Non-Example 3.1.9. Let k be a field and consider the scheme $X = \operatorname{Spec} k[x]/(x^2)$ with its natural morphism $X \to \operatorname{Spec} k$. Then X is irreducible, separated and of finite type over k. However, X is not an algebraic variety over k, since X is not reduced.

For an algebraic scheme, there is a natural characterization of its closed points. For this, we need the following elementary lemma.

Lemma 3.1.10. *Let X be a topological space. Let $W \subset U \subset X$ be subsets equipped with their induced topologies. Let $\overline{W}^U$ be the closure of W in U, and let $\overline{W}^X$ be the closure of W in X. Then $\overline{W}^U = \overline{W}^X \cap U$.*

Proof. As $\overline{W}^X$ is closed in X, we have that $\overline{W}^X \cap U$ is closed in U and contains W. Thus, $\overline{W}^U \subset \overline{W}^X \cap U$. Conversely, we have that $\overline{W}^X$ is the intersection $\cap Z$ of all closed subset $Z \subset X$ that contain W. Hence $\overline{W}^X \cap U$ is contained in the intersection of all closed subset of U that contain W. This gives $\overline{W}^X \cap U \subset \overline{W}^U$. $\square$

Proposition 3.1.11. *Let X be a scheme of finite type over a field k. Let $x \in X$. Then x is closed if and only if there exists an affine open neighbourhood U of $x \in X$ with $\mathcal{O}_X(U)$ a finitely generated k-algebra, with $x \in U$ corresponding to a maximal ideal in $\mathcal{O}_X(U)$. This happens if and only if the residue field $k(x)$ is a finite extension of k.*

Proof. Let $x \in X$ be an arbitrary point. Let $U = \operatorname{Spec} A$ be any affine open neighbourhood of x. If x is closed in X then x is closed in U. Conversely, assume x is closed in U. Define $Z = \overline{x}^X \subset X$. Then Z is irreducible, and hence $Z \cap U$ is irreducible, open and dense in Z. Thus $\dim(Z \cap U) = \dim(Z)$. We have $Z \cap U = \overline{x}^X \cap U = \overline{x}^U$ by Lemma 3.1.10 above. As x is closed in U, we get $Z \cap U = \overline{x}^U = x$. We conclude that $\dim(Z) = \dim(Z \cap U) = \dim(\{x\}) = 0$. Therefore, $Z \subset X$ is an irreducible closed subset of dimension zero, which gives that $\overline{x}^X = x$, i.e. that x is closed in X. We conclude that x is closed in X if and only if x is closed in U.

Now x is closed in $U = \operatorname{Spec} A$ if and only if the prime ideal $x = \mathfrak{p} \in \operatorname{Spec} A$ is a maximal ideal. Thus $k(x) = k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = A/\mathfrak{p}$. This field extension $k(x) \supset k$ is finitely generated as a k-algebra, and therefore finite by the Hilbert Nullstellensatz.

Conversely, assume the residue field $k(x)$ of $x \in U$ is a finite field extension of k, and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. We get ring homomorphisms

$$k \longrightarrow A/\mathfrak{p} \longrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = k(x).$$

Since $k(x)$ is finite as a module over k, we see that $k(x)$ is finite as a module over $A/\mathfrak{p}$. Therefore, we have $\dim(A/\mathfrak{p}) = \dim(k(x)) = 0$. Thus $A/\mathfrak{p}$ is a field, hence $A/\mathfrak{p} = k(x)$. This implies that $\mathfrak{p}$ is a maximal ideal. This proves the proposition. $\square$

Example 3.1.12. Let k be a field and $A = k[t]_{(t)}$ the localization of $k[t]$ in (t). Then A is a one-dimensional local noetherian normal domain, hence a discrete valuation ring (cf. Theorem 3.1.21) with maximal ideal $\mathfrak{m} = (t) \cdot A$. The underlying topological space $|\operatorname{Spec} A|$ consists of two points: $|\operatorname{Spec} A| = \{\eta, \mathfrak{m}\}$. The point $\mathfrak{m}$ is closed and the point $\eta = (0)$ is open. On the one hand, $k(\mathfrak{m}) = k$, which is a finite field extension of k. On the other hand, $k(\eta) = k(t)$, which is not a finite field extension of k.

3.1.3 Smooth varieties

Let k be a field. Let $A = k[t_1, \dots, t_n]/(f_1, \dots, f_m)$ be a finitely generated k-algebra, with $f_i \in k[t_1, \dots, t_n]$ for $i = 1, \dots, m$. Note that for each $i \in \{1, \dots, m\}$ and each $j \in \{1, \dots, n\}$, we get a polynomial

$$\frac{\partial f_i}{\partial t_j} \in k[t_1, \dots, t_n],$$

and hence an element $\frac{\partial f_i}{\partial t_j}(\alpha) \in \bar{k}$ for each $\alpha \in (\bar{k})^n$.

Definition 3.1.13. Fix an integer $d \geq 0$.

(1) Let X be a scheme of finite type over an algebraically closed field k, all whose irreducible components have dimension d. Let $x \in X$ be a closed point (thus $x \in X(k)$, cf. Proposition 3.1.11). We say that X is *smooth at* x if there exists an affine open neighbourhood U of x and an open immersion

$$U \hookrightarrow \operatorname{Spec} k[t_1, \dots, t_n]/(f_1, \dots, f_r)$$

for suitable $n \geq d$, where $r = n - d$, and $f_1, \dots, f_r \in k[t_1, \dots, t_n]$, such that the Jacobian matrix

$$J_x = \left(\frac{\partial f_i}{\partial t_j}(x) \right)_{i,j} \in \mathrm{M}_{r \times n}(k)$$

has rank r.

(2) Let X be a scheme of finite type over a field k, all whose irreducible components have dimension d. Then $X_{\bar{k}}$ is a scheme of finite type over $\bar{k}$, all whose irreducible components have dimension d. Let $x \in X$ be a closed point. We say that X is *smooth at* x if for any closed point $x' \in X_{\bar{k}}$ lying above $x \in X$, $X_{\bar{k}}$ is smooth at x'. We say X is *smooth over* k is X is smooth at every closed point $x \in X$.

Lemma 3.1.14. *Let X be a scheme of finite type over an algebraically closed field k, all whose irreducible components have dimension $d \geq 0$. Let $x \in X(k)$. Assume X is smooth at x. Then for any affine open neighbourhood U of x, any integer $n \geq d$, any $f_1, \dots, f_r \in k[t_1, \dots, t_n]$ where $r = n - d$, and any open immersion*

$$U \hookrightarrow \operatorname{Spec} k[t_1, \dots, t_n]/(f_1, \dots, f_r),$$

the Jacobian matrix

$$J_x = \left(\frac{\partial f_i}{\partial t_j}(x) \right)_{i,j} \in \mathrm{M}_{r \times n}(k)$$

has rank r.

Proof. This follows from results in later chapters (see Exercise 4.2.23). □

Lemma 3.1.15. *Let X be a variety over k. If X is smooth over k then each open subscheme $U \subset X$ is smooth over k.*

Proof. Exercise. □

Example 3.1.16. Let k be a field and let $X = V(F) \subset \mathbb{P}^n_k$ be a hypersurface. Then X is smooth over k if and only if for each

$$\alpha = [x_0 \colon \cdots : x_n] \in X(\bar{k}) \subset \mathbb{P}^n(\bar{k}),$$

there exists $i \in \{0, 1, \ldots, n\}$ such that $(\partial F/\partial x_i)(\alpha) \neq 0$. Indeed, this follows from Lemma 3.1.14. In particular, Definitions 2.4.11 and 3.1.13 are equivalent, if $k = \mathbb{C}$.

Example 3.1.17. Let k be a field and let p be a prime number. Consider the curve $C \subset \mathbb{P}^2_k$ defined by the equation $x_0^p + x_1^p + x_2^p = 0$. In other words,

$$C = \operatorname{Proj}\left(k[x_0, x_1, x_2]/(x_0^p + x_1^p + x_2^p)\right).$$

(1) If the characteristic of k is different from p, then C is smooth. Namely, we have $\partial F/\partial x_i = p \cdot x_i^{p-1}$ for $i = 0, 1, 2$, and if, for each $i \in \{0, 1, 2\}$, this homogeneous degree $p - 1$ polynomial $p \cdot x_i^{p-1}$ vanishes at some $\alpha = [a_0 \colon a_1 \colon a_2] \in \mathbb{P}^2(\bar{k})$, then $a_0 = a_1 = a_2 = 0$, which is absurd.

(2) If the characteristic of k equals p, then C is not smooth. Namely, we then have $\partial F/\partial x_i = p \cdot x_i^{p-1} = 0$ for $i = 0, 1, 2$. Thus for *any* $\alpha \in C(\bar{k})$, we get $F(\alpha) = \partial F/\partial x_i(\alpha) = 0$ for $i = 0, 1, 2$.

3.1.4 Normal schemes

We consider the following important notion in scheme theory.

Definition 3.1.18. (1) Let A be a ring which is a domain. Then A is called *normal* if A is integrally closed in its field of fractions $Q(A)$. This means that for each $\alpha \in Q(A)$ which is integral over A, we have $\alpha \in A$. Equivalently: for each monic polynomial $f \in A[x]$ and each $\alpha \in Q(A)$ with $f(\alpha) = 0$, we have $\alpha \in A$.

(2) A ring R is *normal* if for each prime ideal $\mathfrak{p} \subset R$, the localization $R_{\mathfrak{p}}$ is a normal domain.

(3) A scheme X is called *normal* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

Suppose $X = \operatorname{Spec} A$ is an affine scheme such that A is reduced. Then saying that X is normal is not equivalent to saying that A is integrally closed in its total ring of fractions. However, if A is noetherian, then this is the case (exercise).

Lemma 3.1.19. *Let X be a scheme. The following are equivalent:*

(1) The scheme X is normal.

(2) For every affine open $U \subset X$, the ring $\mathcal{O}_X(U)$ is normal.

(3) *There exists an affine open covering $X = \cup_i U_i$ such that each ring $\mathcal{O}_X(U_i)$ is normal.*

(4) *There exists an open covering $X = \cup_i X_i$ such that the scheme X_i is normal for each i.*

Moreover, if X is normal, then every open subscheme $U \subset X$ is normal.

Proof. Exercise. □

Lemma 3.1.20. *Let X be a normal integral scheme. Then for each non-empty open $U \subset X$, the scheme U is normal and integral, and $\mathcal{O}_X(U)$ is a normal integral domain.*

Proof. The fact that U is normal and integral is clear. Thus, it suffices to show that $\mathcal{O}_X(X)$ is a normal integral domain. For this, see e.g. [Stacks Project, tag 0358]. □

Theorem 3.1.21. *Let A be a noetherian local domain of dimension one, with maximal ideal $\mathfrak{m}$. The following are equivalent:*

(1) *A is a discrete valuation ring;*

(2) *A is normal;*

(3) *$\mathfrak{m}$ is a principal ideal.*

Proof. See Atiyah–Maconald (Proposition 9.2 on page 94). □

Corollary 3.1.22. *Let k be an algebraically closed field and let C be a curve over k. Then C is smooth over k if and only if C is normal.*

Proof. This uses: (1) any discrete valuation ring is a *regular* local ring of dimension one, and conversely; (2) since k is algebraically closed, any variety X over k is smooth over k if and only if for each $x \in X$ there exists an affine open neighbourhood $U \subset X$ such that the localizations $R_\mathfrak{p}$ of $R = \mathcal{O}_X(U)$ are all regular. Details omitted. □

In arbitrary dimensions, one has:

Proposition 3.1.23. *Let X be a smooth variety over a field k. Then X is normal.*

Proof. We do not prove this here. □

3.1.5 Codimension

Definition 3.1.24. Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset of X. The *codimension* of Y in X, denoted by $\operatorname{codim}(Y, X)$, is the supremum of all integers n such that there exists a chain

$$Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$$

of irreducible closed subsets Y_i of X.

Proposition 3.1.25. *Let X be a scheme, let $x \in X$ and define $Y = \overline{\{x\}} \subset X$. Then Y is irreducible, and* $\operatorname{codim}(Y, X) = \dim \mathcal{O}_{X,x}$.

Proof. Since Y has a generic point, it is irreducible. Let $Y = Y_0 \subsetneq \cdots \subsetneq Y_n \subset X$ be a chain of irreducible closed subsets. Let $U \subset X$ be an affine open neighbourhood of x in X. Since $U \cap Y_i \neq \emptyset$ for each i, we have $\eta_i \in U$ for each i. Moreover, for each i, $Y_i \cap U$ is a closed subset in U, defined by a prime ideal $\mathfrak{p}_i \subset R$, where $R = \mathcal{O}_X(U)$. Thus we get a chain of prime ideals

$$\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_0 = \mathfrak{p},$$

where $\mathfrak{p}$ is the prime ideal that defines $Y \cap U$ in U. Hence we have

$$\operatorname{codim}(Y, X) = \sup_n (\exists \mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_0 = \mathfrak{p} \subset R) = \operatorname{height}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}).$$

As $R_{\mathfrak{p}} = \mathcal{O}_{X,x}$, we get $\dim \mathcal{O}_{X,x} = \dim R_{\mathfrak{p}} = \operatorname{codim}(Y, X)$, whence the result. □

Theorem 3.1.26. *Let k be a field and let X be a variety over k, with generic point $\eta \in X$. Let $k(X) = \mathcal{O}_{X,\eta}$ be the function field of X. Then:*

(1) the dimension of X agrees with the transcendence degree of $k(X)$ over k;

(2) for each non-empty open subset $U \subset X$, we have $\dim(U) = \dim(X)$*;*

(3) if $Y \subset X$ is a closed subvariety, then all maximal chains of irreducible subvarieties

$$Y \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n \subset X$$

have the same length;

(4) we have $\operatorname{codim}(Y, X) = \dim(X) - \dim(Y)$.

Proof. We will not prove this here. □

3.1.6 Weil divisors

Definition 3.1.27. Let X be a normal integral noetherian scheme.

(1) A *prime divisor* is an integral subscheme $Z \subset X$ of codimension one.

(2) A *Weil divisor* of X is an element of the free abelian group generated by the prime divisors of X. We denote the group of Weil divisors by $\operatorname{Div}(X)$. Thus, an element $D \in \operatorname{Div}(X)$ can be written as a formal linear combination of prime divisors

$$D = \sum_{Z \subset X \text{prime}} n_Z \cdot Z$$

with $n_Z \in \mathbb{Z}$ for each prime divisor $Z \subset X$, and such that $n_Z = 0$ for all but finitely many prime divisors $Z \subset X$.

(3) We say that a Weil divisor $D = \sum n_Z \cdot Z$ is *effective* if $n_Z \geq 0$ for each prime divisor Z.

(4) Any Weil divisor $D = \sum n_Z Z$ can be written as $D = \sum_{i=1}^k n_i \cdot Z_i$ where Z_i is a prime divisor and $n_i \in \mathbb{Z} - \{0\}$ for each $i \in \{1, \ldots, k\}$. This gives a closed subset $\cup_i Z_i \subset X$ called the *support* of the Weil divisor D.

(5) Given two Weil divisors $D = \sum_Z n_Z Z$ and $D' = \sum m_Z Z$, we say that $D \geq D'$ if $D - D'$ is effective, or equivalently, if $n_Z \geq m_Z$ for all prime divisors Z. This turns $\mathrm{Div}(X)$ into a partially ordered group.

Example 3.1.28. Let k be a field and let $X = \mathbb{P}^1_k$ be the projective line over k. Since C is a curve, any irreducible closed subset of codimension one on X is a closed point. For example, for any

$$f \in \mathrm{Hom}_{\mathsf{Sch}/k}(\mathrm{Spec}\ k, \mathbb{P}^1_k) = \mathbb{P}^1_k(k) = \left\{\text{lines in } k^2\right\},$$

the image $f(\mathrm{Spec}\ k)$ in $\mathbb{P}^1_k$ is a closed point (see Proposition 3.1.11), and the map

$$\mathbb{P}^1_k(k) \to \left\{\text{closed points } x \in \mathbb{P}^1_k\right\}$$

is injective. In this way, we get some examples of Weil divisors on $\mathbb{P}^1_k$:

$$\begin{aligned} D_1 &:= 3 \cdot (1\colon 0) - 5 \cdot (0\colon 1), \\ D_2 &:= (1\colon 1) + 5 \cdot (0\colon 1), \\ D_1 + D_2 &= 3 \cdot (1\colon 0) + (1\colon 1). \end{aligned}$$

3.2 Lecture 21 : The divisor class group of a scheme

3.2.1 *Principal Weil divisors*

Let X be a normal integral noetherian scheme with generic point $\eta \in X$ and fraction field $K = k(X) = \mathcal{O}_{X,\eta}$. Since X is normal, for each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a domain which is integrally closed in its field of fractions $Q(\mathcal{O}_{X,x}) = K$.

Lemma 3.2.1. *Let X be a normal integral noetherian scheme. Let $\xi \in X$ be a point such that* $\mathrm{codim}(\overline{\{\xi\}}, X) = 1$.

(1) The reduced closed subscheme $\overline{\{\xi\}} \subset X$ is a prime divisor, and every prime divisor arises uniquely in this way.

(2) The local ring $A = \mathcal{O}_{X,\xi}$ is a discrete valuation ring.

Proof. Note that $\overline{\{\xi\}}$ is irreducible since it has a generic point, hence it is a prime divisor. For an arbitrary prime divisor $Z \subset X$, the generic point η_Z of Z gives a codimension one point $\eta_Z \in X$. As for part (2), this follows from Theorem 3.1.21. □

This has the following implication. By Theorem 3.1.21, for each codimension one point $\xi \in X$, the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring. Thus, this ring is equipped with an associated valuation

$$v\colon K \longrightarrow \mathbb{Z} \cup \{\infty\},$$

such that $A = v^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\})$.

In fact, one can define v explicitly as follows. Given $a \in A - \{0\}$, the ideal $(a) \subset A$ has the property that $(a) = \mathfrak{m}^n$ for some $n \in \mathbb{Z}_{\geq 0}$, and we define $v(a) = n$. This gives a function $v\colon A - \{0\} \to \mathbb{Z}$ which extends to $K^* = \left\{\frac{a}{b}\colon a, b \in A - \{0\}\right\}$ by putting $v(a/b) = v(a) - v(b)$, and then to a map $v\colon K \to \mathbb{Z} \cup \{\infty\}$ by putting $v(0) = \infty$.

Definition 3.2.2. Let $f \in K = k(X)$. For every prime divisor $Z \subset X$, we get by the above a valuation $v_Z\colon K \to \mathbb{Z} \cup \{\infty\}$, which allows us to define

$$\operatorname{ord}_{Z,X}(f) \coloneqq v(f).$$

Lemma 3.2.3 (Algebraic Hartog's lemma)**.** *Let A be a normal noetherian integral domain and let $x \in K$. Let $K = Q(A) = \operatorname{Frac}(A)$ be the fraction field of A. Then $x \in A$ if and only if $x \in A_{\mathfrak{p}} \subset K$ for all height one primes ideals $\mathfrak{p} \subset A$.*

Proof. We do not prove this here. □

Corollary 3.2.4. *Let A be a noetherian normal domain, and $f \in Q(A)$. Then $\operatorname{ord}_{V(\mathfrak{p}),\operatorname{Spec} A}(f) \geq 0$ for all primes $\mathfrak{p} \subset A$ of height one if and only if $f \in A$, and $\operatorname{ord}_{V(\mathfrak{p}),\operatorname{Spec} A}(f) = 0$ for all primes $\mathfrak{p} \subset A$ of height one if and only if $f \in A^*$.*

Proof. Let $f \in Q(A)^*$. Then apply Lemma 3.2.3 to f and to $f^{-1} \in Q(A)$. □

Lemma 3.2.5. *Suppose that X is a normal integral noetherian scheme with fraction field K and let $f \in K^*$. Then $\operatorname{ord}_{Z,X}(f) = 0$ for all but finitely many primes $Z \subset X$.*

Proof. We proceed in two steps:

Step 1: *Reduction to the case where $X = \operatorname{Spec} A$ is affine and $f \in A$:* Consider a non-empty affine open subset V of X. Let $R = \mathcal{O}_X(V)$. Then K is the fraction field of R, so that $f = a/b$ for some $a, b \in R$ which are both non-zero. We then look at the affine open $U \coloneqq D(b) \subset V \subset X$. This is an affine open where b is invertible, so that $f = a/b \in R_b = \Gamma(U, \mathcal{O}_X)$. The complement $W \coloneqq X - U$ is a closed subset of codimension at least one, since X is integral (which implies U is non-empty). Notice that

$$\sum_Z \operatorname{ord}_{Z,X}(f) \cdot Z = \sum_{Z \subset W} \operatorname{ord}_{Z,X}(f) Z + \sum_{Z \not\subset W} \operatorname{ord}_{Z,X}(f) Z,$$

and that there are only finitely many prime divisors $Z \subset X$ that satisfy $Z \subset W$. Thus, it suffices to show that $\operatorname{ord}_Z(f) = 0$ for almost all prime divisors $Z \subset X$ with $Z \cap U \neq \emptyset$. Notice that, for primes $Z \subset X$ with $Z \cap U \neq \emptyset$, we have

$$\operatorname{ord}_{Z,X}(f) = \operatorname{ord}_{Z \cap U, U}(f),$$

since $\mathcal{O}_{X,\xi} = \mathcal{O}_{U,\xi}$ for the generic point $\xi \in Z$. Now the sum $\sum_{Z \subset W} \mathrm{ord}_{Z,X}(f)Z$ is finite since W has finitely many irreducible components of codimension one. Hence it remains to show that $\mathrm{ord}_{Z\cap U,U}(f) = 0$ for $f \in \Gamma(U, \mathcal{O}_X)$ and almost all primes $Z \subset X$ with $Z \cap U \neq \emptyset$, so that indeed, we may assume that $X = \mathrm{Spec}\ A$ is affine and $f \in A$.

Step 2: *Case where* $X = \mathrm{Spec}\ A$ *is affine and* $f \in A$*:* We now have $\mathrm{ord}_Z(f) \geq 0$, and $\mathrm{ord}_Z(f) > 0$ if and only if $\mathfrak{p}|(f)_\mathfrak{p} \subset A_\mathfrak{p}$ for all $\mathfrak{p}$ of height one in Z if and only if $f \in \mathfrak{p}$ for all primes $\mathfrak{p}$ in Z if and only if Z is contained in $V(f) \subset \mathrm{Spec}\ A$. Since $V(f)$ has finitely many irreducible components of codimension one, we are done. □

Definition 3.2.6. Let X be a normal integral noetherian scheme with fraction field K. For $f \in K^*$, define its corresponding Weil divisor $\mathrm{div}(f)$ as

$$\mathrm{div}(f) \coloneqq \sum_Z \mathrm{ord}_{Z,X}(f) \cdot Z,$$

where the sum runs over all prime divisors. Any Weil divisor D of the form $D = \mathrm{div}(f)$ for some $f \in K^*$ is called a *principal* Weil divisor.

Example 3.2.7. Let A be a normal noetherian integral domain and let $X = \mathrm{Spec}\ A$. Let K be the fraction field of A. Then for any $f \in K^*$, we have

$$\mathrm{div}(f) = \sum_{\mathfrak{p} \text{ height } 1} \mathrm{ord}_{V(\mathfrak{p}),\mathrm{Spec}\ A}(f) \cdot V(\mathfrak{p}).$$

Example 3.2.8. Let A be a discrete valuation ring with maximal ideal $\mathfrak{m} \subset A$. Let $t \in A$ such that $\mathfrak{m} = (t) \subset A$. The underlying topological space $|\mathrm{Spec}\ A|$ consists of two points: $|\mathrm{Spec}\ A| = \{\eta, \mathfrak{m}\}$. The point $\mathfrak{m}$ is closed and the point $\eta = (0)$ is open. We have $(0) \subsetneq \mathfrak{m}$ and $\mathfrak{m}$ is the only prime ideal of height one. For $f \in K = \mathrm{Frac}(A)$, we can write $f = u \cdot t^n$ for some $n \in \mathbb{Z}$ and $u \in A^*$. Then $\mathrm{div}(f) = \mathrm{ord}_{V(\mathfrak{m}),\mathrm{Spec}\ A}(f)$ and this equals $v(f) = v(u \cdot t^n) = v(t^n) = n$, where $v\colon K \to \mathbb{Z} \cup \{\infty\}$ is the valuation.

Lemma 3.2.9. *Let* X *be a normal integral noetherian scheme. The set of principal Weil divisors forms a subgroup of* $\mathrm{Div}(X)$*.*

Proof. For $f, g \in K^*$, we have $\mathrm{div}(f) - \mathrm{div}(g) = \mathrm{div}(f/g)$. □

In fact, the map $K^* \to \mathrm{Div}(X)$ sending f to $\mathrm{div}(f)$, is a group homomorphism. If $X = \mathrm{Spec}\ A$ is affine, then $\mathrm{div}(f) = 0$ if and only if $f \in A^*$ (see Corollary 3.2.4); thus we get an exact sequence $0 \to A^* \to K^* \to \mathrm{Div}(X)$ in that case.

3.2.2 Examples

Example 3.2.10. Let $X = \mathrm{Spec}\ \mathbb{Z}$ with function field $Q(\mathbb{Z}) = \mathbb{Q}$. We claim that the map $\mathbb{Q}^* \to \mathrm{Div}(X)$ is surjective. Indeed, any element $D \in \mathrm{Div}(X)$ is a finite sum $D = \sum_i n_i \cdot V(p_i)$, where the p_i are prime numbers and $n_i \in \mathbb{Z}$; we have $\mathrm{div}(\prod_i p_i^{n_i}) = D$.

Example 3.2.11. Let $X = \mathbb{A}^1_k$. Consider $f = t^2(t-1)^{-1} \in k(t) = k(\mathbb{A}^1_k)$. Then $\mathrm{div}(f) = 2 \cdot [0] - [1]$, where $0, 1 \in \mathbb{A}^1(k)$ give closed points of $\mathbb{A}^1_k$.

Example 3.2.12. Let k be a field and consider $X = \mathbb{P}^1_k = \operatorname{Proj}(k[x_0, x_1])$, whose function field is $k(X) = k(t)$, where $t = x_1/x_0$. Consider the rational function

$$f = t^2(t-1)^{-1} \in K.$$

Notice that $\mathbb{P}^1_k - U_0 = \{\infty\}$, where $U_0 = D_+(x_0) = \operatorname{Spec} k[x_0, x_1]_{(x_0)} = \operatorname{Spec} k[t]$, and where $\infty = [0 : 1] \in U_1(k)$. Therefore:

$$\begin{aligned}\operatorname{div}(f) &= \sum_{p \in U_0} \operatorname{ord}_p(f) + \operatorname{ord}_\infty(f) \cdot \infty \\ &= 2 \cdot [1 \colon 0] - [1 \colon 1] + \operatorname{ord}_\infty(f) \cdot \infty,\end{aligned}$$

because

$$\sum_{p \in U_0} \operatorname{ord}_p(f) = \sum_{p \in \operatorname{Spec} k[t]} \operatorname{ord}_p(f) = 2 \cdot [0] - [1]$$

by Example 3.2.11. Moreover, using the identification

$$U_1 = D_+(x_1) = \operatorname{Spec} k[x_0, x_1]_{(x_1)} = \operatorname{Spec} k[u]$$

with $u = x_0/x_1 = t^{-1}$, we get

$$f = t^2(t-1)^{-1} = u^{-2}(u^{-1}-1)^{-1} = \frac{1}{u^2(u^{-1}-1)} = \frac{1}{u-u^2}.$$

Therefore, if we let $g = (u - u^2)^{-1} = u^{-1}(1-u)^{-1} \in k(u)$, then

$$\operatorname{ord}_\infty(f) = \operatorname{ord}_0(g) = -1.$$

All in all, this gives

$$\operatorname{div}(f) = \sum_{p \in U_0} \operatorname{ord}_p(f) + \operatorname{ord}_\infty(f) \cdot [0 \colon 1] = 2 \cdot [1 \colon 0] - [1 \colon 1] - [0 \colon 1].$$

3.2.3 *The divisor class group*

Definition 3.2.13. Let X be a noetherian integral normal scheme with function field K. We define the *divisor class group* of X (or simply the *class group* of X) as the group of Weil divisors modulo principal Weil divisors, and we denote it by $\operatorname{Cl}(X)$. Thus, we have

$$\operatorname{Cl}(X) = \operatorname{Div}(X) / \langle \operatorname{div}(f) \mid f \in K^* \rangle.$$

Two Weil divisors D and D' are said to be *linearly equivalent* (written $D \sim D'$) if they have the same image in $\operatorname{Cl}(X)$; in other words, if $D - D' = \operatorname{div}(f)$ for some $f \in K^*$.

Example 3.2.14. Let A be a noetherian normal domain with fraction field K. Write $\operatorname{Div}(A) = \operatorname{Div}(\operatorname{Spec} A)$ and $\operatorname{Cl}(A) = \operatorname{Cl}(\operatorname{Spec} A)$. In view of Corollary 3.2.4, there is an exact sequence of abelian groups

$$0 \longrightarrow A^* \longrightarrow K^* \longrightarrow \operatorname{Div}(A) \longrightarrow \operatorname{Cl}(A) \longrightarrow 0. \tag{3.2}$$

Remark 3.2.15. Let K be a number field. Then K is the fraction field of its ring of integers $\mathcal{O}_K$, and in this case, $\mathrm{Div}(\mathcal{O}_K)$ can be identified with the group of *fractional ideals* (these are non-zero finitely generated $\mathcal{O}_K$-submodules of K, which form a group under ideal multiplication), and $\mathrm{Cl}(\mathcal{O}_K)$ with the group of fractional ideals modulo the *principal fractional ideals* (these are the fractional ideals generated by an element of K^*). A classical result in number theory says that the group $\mathrm{Cl}(\mathcal{O}_K)$ is finite. Note that $\mathrm{Cl}(\mathcal{O}_K) = 0$ if and only if $\mathcal{O}_K$ is a unique factorization domain. For example, $\mathbb{Z}[\sqrt{-5}]$ is not a UFD since $2 \cdot 3 = (1-\sqrt{-5})(1+\sqrt{-5})$, and in fact $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2$.

Example 3.2.16. Consider the ring $\mathbb{Z}$. Then $\mathrm{Cl}(\mathbb{Z}) = 0$, see Example 3.2.10.

This generalizes as follows:

Proposition 3.2.17. *Let A be a normal noetherian integral domain and let $X = \operatorname{Spec} A$. Then $\mathrm{Cl}(X) = 0$ if and only if A is a unique factorization domain.*

Proof. Suppose that A is a unique factorization domain. Let $Z \subset X$ be a non-zero prime divisor in X. Then $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$ of height one. Take $f \in \mathfrak{p}$ non-zero, and let $f = f_1 \cdots f_n$ be a factorization of f into irreducible elements of A. Since $\mathfrak{p}$ is prime, we see that $f_i \in \mathfrak{p}$ for some i. Since A is a UFD, the element f_i is prime. Thus $\mathfrak{p}$ contains the prime ideal (f_i). As $\mathfrak{p}$ has height one, we have $\mathfrak{p} = (f_i)$. Thus gives $Z = V(\mathfrak{p}) = V(f_i) \subset X$. But note that $\mathrm{div}(f) = V(f_i)$. Therefore, $Z = \mathrm{div}(f_i)$, and we get that $\mathrm{Cl}(X) = 0$.

Conversely, assume $\mathrm{Cl}(X) = 0$. Then every height one prime ideal $\mathfrak{p}$ is principal. Indeed, there is an $f \in K^*$ such that $\mathrm{div}(f) = V(\mathfrak{p})$, one has $f \in A$ (in view of the exact sequence (3.2)), and one can show that $\mathfrak{p} = (f)$ (exercise). Now since A is noetherian, every non-zero non-unit element $a \in A$ has a factorization into irreducibles, hence it suffices to show that an irreducible element $a \in A$ is prime. Let $(a) \subset \mathfrak{p}$ be a minimal prime over (a). Then $\mathfrak{p}$ has height one (exercise). By the above, $\mathfrak{p}$ is principal, so that $\mathfrak{p} = (b)$ for some $b \in A$. Hence $a \in (b)$ so that $a = bc$ for some $c \in A$, which must be a unit because a is irreducible. Thus, $(a) = (b) = \mathfrak{p}$ is prime, and we win. $\square$

Corollary 3.2.18. *Let k be a field and let $n \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{Cl}(\mathbb{A}^n_k) = 0$.* $\square$

3.3 Lecture 22 : Weil divisors and invertible sheaves

3.3.1 Class group of projective space

Let k be a field and consider $\mathbb{P}^n_k = \mathrm{Proj}(R)$ with $R = k[x_0, \dots, x_n]$. Prime divisors Z on $\mathbb{P}^n_k$ are of the form $Z = V(\mathfrak{p})$ for a non-zero homogeneous height one prime ideal $\mathfrak{p} \subset R$. For such a prime ideal $\mathfrak{p}$ we have $\mathfrak{p} = (g)$ for some non-zero irreducible homogeneous polynomial $g \in R$ (see the proof of Proposition 3.2.17). The generator g is unique up to scalar, so the degree $\deg(\mathfrak{p}) := \deg(g)$ of a homogeneous height one prime ideal $\mathfrak{p}$ is well-defined. This gives a group homomorphism

$$\deg\colon \mathrm{Div}(\mathbb{P}^n_k) \longrightarrow \mathbb{Z}, \qquad \sum_{i=1}^k n_i V(\mathfrak{p}_i) \mapsto \sum_{i=1}^k n_i \deg(\mathfrak{p}_i).$$

Exercise 3.3.1. Let k be a field.

(1) For a rational function $f \in K(\mathbb{P}^n_k)$, show that $\deg(\operatorname{div}(f)) = 0$.

(2) Show that deg factors through an isomorphism

$$\operatorname{Cl}(\mathbb{P}^n_k) \xrightarrow{\sim} \mathbb{Z},$$

and compare this statement with Exercise 2.3.11.

3.3.2 The sheaf associated to a Weil divisor

Definition 3.3.2. Let X be a normal integral noetherian scheme with function field K, and let $D = \sum n_Z Z$ be a Weil divisor on X. We define a presheaf $\mathcal{O}_X(D)$ by defining, for $U \subset X$ an open subset,

$$\mathcal{O}_X(U) := \{f \in K \mid \operatorname{ord}_{Z,X}(f) \geq -n_Z \text{ for all } Z \text{ with generic point } \eta_Z \in U\}.$$

Exercise 3.3.3. Check that this presheaf $\mathcal{O}_X(D)$ is actually a sheaf. As such, it is a subsheaf of the constant sheaf of fields $\mathcal{K}$ on X, associated to the field K. Finally, verify that $\mathcal{O}_X(D)$ has a natural $\mathcal{O}_X$-module structure.

Proposition 3.3.4. *The $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ is quasi-coherent.*

Proof. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset. The proposition follows from the fact that for $f \in A$, the canonical injective map

$$\Gamma(U, \mathcal{O}_X(D))_f \longrightarrow \Gamma(D(f), \mathcal{O}_X(D))$$

induced by restriction to the open subset $D(f) \subset U$, is an isomorphism. We leave this fact as an exercise for the reader. □

Lemma 3.3.5. *Let X be a normal integral noetherian scheme with function field K.*

(1) For each non-empty open $U \subset X$, the ring $\mathcal{O}_X(U)$ is a normal integral domain.

(2) For each non-empty open $U \subset X$, we have that $\eta \in U$ for the generic point η of X, and the natural map

$$\varphi \colon \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\eta} = K$$

is injective. This gives an embedding of sheaves $\mathcal{O}_X \hookrightarrow \mathcal{K}$.

(3) For the Weil divisor $D = 0$, we have $\mathcal{O}_X = \mathcal{O}_X(D)$ as subsheaves of $\mathcal{K}$. Conversely, if for some Weil divisor D on X, $\mathcal{O}_X(D) = \mathcal{O}_X \subset \mathcal{K}$, then $D = 0$.

(4) For Weil divisors D, E on X, we have $\mathcal{O}_X(D) = \mathcal{O}_X(E)$ as subsheaves of $\mathcal{K}$ if and only if $D = E$.

Proof. For part (1), see Lemma 3.1.20.

For part (2): let $f \in \mathcal{O}_X(U)$ such that $\varphi(f) = 0$. We claim that $f = 0$. To prove this, consider a cover $U = \cup_i U_i$ by affine opens U_i. As it suffices to show that $f|_{U_i} = 0$ for each i, we may assume that $U = \operatorname{Spec} A$ is affine. Then $f \in A$, and $K = Q(A)$ is the fraction field of A. Since A is an integral domain (see Lemma 3.1.20), the map $A \to Q(A) = K$ is injective. Hence, $f = 0$ as desired.

As for part (3), consider a non-empty open $U \subset X$. We have

$$\mathcal{O}_X(0)(U) = \{f \in K \mid \operatorname{ord}_{Z,X}(f) \geq 0 \text{ for all } Z \text{ prime with generic point } \eta_Z \in U\}.$$

This implies that

$$\mathcal{O}_X \subset \mathcal{O}_X(0) \subset \mathcal{K}$$

as subsheaves of $\mathcal{K}$. To prove that the inclusion $\mathcal{O}_X \subset \mathcal{O}_X(0)$ is an equality of subsheaves of $\mathcal{K}$, it suffices to show that $\mathcal{O}_X(U) = \mathcal{O}_X(0)(U) \subset K$ for each non-empty affine open $U \subset X$. If $U = \operatorname{Spec} A$ for a normal noetherian integral domain A, then this follows from the fact that

$$A = \left\{f \in K \mid \operatorname{ord}_{V(\mathfrak{p}),U}(f) \geq 0 \text{ for all height one } \mathfrak{p} \in U\right\},$$

see Corollary 3.2.4.

Conversely, let D be a Weil divisor so that $\mathcal{O}_X(D) = \mathcal{O}_X \subset \mathcal{K}$. Write $D = \sum n_Z Z$. Assume that $n_Z \neq 0$ for some prime divisor $Z \subset X$; our goal is to arrive at a contradiction. We may assume that $n_Z > 0$. Let $z \in Z$ be the generic point of Z, and let $U \subset X$ be an affine open neighbourhood of z in X. Let $A = \mathcal{O}_X(U)$, and let $\mathfrak{p} \subset A$ be the height one prime ideal corresponding to $z \in U$. Then $\mathcal{O}_{U,z} = A_{\mathfrak{p}}$, and this is a discrete valuation ring. Moreover, we have $Z \cap U = V(\mathfrak{p}) \subset U = \operatorname{Spec} A$, and

$$\mathcal{O}_X(D)_z = \left\{f \in K \mid \operatorname{ord}_{V(\mathfrak{p}),U}(f) \geq -n_Z\right\}.$$

As we assume that $\mathcal{O}_X(D) = \mathcal{O}_X$, we get $\mathcal{O}_X(D)_z = \mathcal{O}_{X,z} = A_{\mathfrak{p}}$, hence

$$\left\{f \in K \mid \operatorname{ord}_{V(\mathfrak{p}),U}(f) \geq -n_Z\right\} = \mathcal{O}_X(D)_z = A_{\mathfrak{p}} = \left\{f \in K \mid \operatorname{ord}_{V(\mathfrak{p}),U}(f) \geq 0\right\}. \quad (3.3)$$

We claim this is a contradiction. Indeed, $A_{\mathfrak{p}}$ is a discrete valuation ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, which is generated by a single element $t \in \mathfrak{p}A_{\mathfrak{p}}$, see Theorem 3.1.21. This gives an element

$$f := t^{-1} \in K$$

that has the property that

$$\operatorname{ord}_{V(\mathfrak{p}),U}(f) = v(t^{-1}) = -1.$$

Hence $f \in \mathcal{O}_X(D)_z$ (since $n_Z > 0$) but $f \notin \mathcal{O}_{X,z}$, which violates (3.3).

Finally, to prove part (4), we apply part (3) to the Weil divisor $D - E$. The fact that $\mathcal{O}_X(D) = \mathcal{O}_X(E)$ implies that $\mathcal{O}_X(D - E) = \mathcal{O}_X \subset \mathcal{K}$, so that $D = E$. □

Lemma 3.3.6. *Let X be a normal integral noetherian scheme with function field K. Let D be a Weil divisor on X.*

(1) If $f \in K$, then

$$\mathcal{O}_X(\operatorname{div}(f)) = f^{-1} \cdot \mathcal{O}_X \subset \mathcal{K} \tag{3.4}$$

as subsheaves of the constant sheaf $\mathcal{K}$ associated to K.

(2) Then D is a principal divisor if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

Proof. For item (1), let $U \subset X$ be a non-empty affine open with $U = \operatorname{Spec} A$. We claim that

$$\Gamma(U, \mathcal{O}_X(\operatorname{div}(f))) = f^{-1} \cdot A \subset K. \tag{3.5}$$

Indeed,

$$\Gamma(U, \mathcal{O}_X(\operatorname{div}(f))) = \left\{g \in K \mid \operatorname{ord}_{V(\mathfrak{p}),U}(g) \geq -\operatorname{ord}_{V(\mathfrak{p}),U}(f) \text{ for all } \mathfrak{p} \text{ of height one}\right\},$$

so that

$$f \cdot \Gamma(U, \mathcal{O}_X(\operatorname{div}(f))) = \left\{g \in K \mid \operatorname{ord}_{V(\mathfrak{p}),U}(g) \geq 0 \text{ for all primes } \mathfrak{p} \text{ of height one}\right\} = A,$$

where the second equality holds by Corollary 3.2.4. Hence (3.5) follows, proving (3.4).

For item (2), assume that $D = \operatorname{div}(f)$ for some $f \in K^*$. Then $\mathcal{O}_X(\operatorname{div}(f)) = f^{-1} \cdot \mathcal{O}_X$ by item (1), and multiplication by $f \in K^*$ defines an isomorphism $f^{-1} \cdot \mathcal{O}_X \cong \mathcal{O}_X$.

Conversely, assume that $D = \sum n_Z Z$ is a Weil divisor such that $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Since $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-submodule of $\mathcal{K}$, the fact that it is free of rank one over $\mathcal{O}_X$ implies that there exists $g \in K^*$ such that $\mathcal{O}_X(D) = \mathcal{O}_X \cdot g \subset \mathcal{K}$. Define $f = g^{-1}$. We claim that $D = \operatorname{div}(f)$. Note that

$$\mathcal{O}_X(\operatorname{div}(f)) = f^{-1} \cdot \mathcal{O}_X = g \cdot \mathcal{O}_X \subset \mathcal{K}$$

by item (1). Therefore,

$$\mathcal{O}_X(D) = \mathcal{O}_X \cdot g = \mathcal{O}_X(\operatorname{div}(f)) \subset \mathcal{K}$$

as subsheaves of $\mathcal{K}$. Thus, we have $D = \operatorname{div}(f)$ by item (4) of Lemma 3.3.5. □

Definition 3.3.7. Let X be a normal integral noetherian scheme. Let D be a Weil divisor on X. For a non-empty open subscheme U, U is a normal integral noetherian scheme, see Lemma 3.1.20. We define a Weil divisor $D|_U$ on U as follows: if $D = \sum_{i=1}^k n_i Z_i$ for some $n_i \in \mathbb{Z}$ and prime divisors $Z_i \subset X$, we let $J \subset \{1, \dots, k\}$ be the subset of those $j \in \{1, \dots, k\}$ such that $Z_j \cap U \neq \emptyset$ (equivalently, such that the generic point η_j of Z_j is contained in U). We then define $D|_U = \sum_{j \in J} n_j (Z_j \cap U)$. The fact that $Z_j \cap U \subset U$ is a prime divisor (whenever $Z_j \cap U \neq \emptyset$) follows from the fact that $\operatorname{codim}(Z_j \cap U, U) = \dim \mathcal{O}_{U,\eta_j} = \mathcal{O}_{X,\eta_j} = \operatorname{codim}(Z_j, X) = 1$, see Lemma 3.1.25.

Corollary 3.3.8. *Let X be a normal integral noetherian scheme. Let D be a Weil divisor on X. Then the following are equivalent:*

(1) The $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ is invertible (i.e. locally free of rank one).

(2) The Weil divisor D is locally principal*; that is, there exists an open covering $X = \cup U_i$ of X and rational functions $f_i \in K^*$ such that $D|_{U_i} = \operatorname{div}(f_i)$.*

Proof. Let $U \subset X$ be a non-empty open subscheme. Then U is a normal integral noetherian scheme, see Lemma 3.1.20, and $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U)$. Moreover, by item (2) of Lemma 3.3.6, we have that $\mathcal{O}_U(D|_U) \cong \mathcal{O}_U$ if and only if $D|_U$ is a principal divisor. Consequently, we see that the $\mathcal{O}_U$-module $\mathcal{O}_X(D)|_U$ is trivial (i.e. isomorphic to $\mathcal{O}_U$) if and only if $D|_U$ is a principal divisor. In particular, the sheaf $\mathcal{O}_X(D)$ is locally free of rank one if and only if the Weil divisor D is locally principal. □

Exercise 3.3.9. Let X be a normal integral noetherian scheme. Let $D \subset X$ be an integral closed subscheme of codimension one, and $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of D in X.

(1) Show that the subsheaf $\mathcal{O}_X(-D) \subset \mathcal{K}$ is contained in $\mathcal{O}_X \subset \mathcal{K}$.

(2) Show that $\mathcal{I} = \mathcal{O}_X(-D)$ as subsheaves of $\mathcal{O}_X$.

3.3.3 Cartier divisors

Definition 3.3.10. Let X denote a normal integral noetherian scheme with function field K and sheaf of rational functions $\mathcal{K}$. Consider the exact sequence of sheaves of abelian groups:

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}^* \longrightarrow \mathcal{K}^*/\mathcal{O}_X^* \longrightarrow 0.$$

It induces a short exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*).$$

(1) A *Cartier divisor* on X is a global section $D \in \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$ of the sheaf $\mathcal{K}^*/\mathcal{O}_X^*$.

(2) We define

$$\operatorname{CaDiv}(X) \coloneqq \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$$

as the abelian group of Cartier divisors.

(3) A Cartier divisor is *principal* if it is in the image of $K^* = \Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$.

(4) Two Cartier divisors are called *linearly equivalent* if their difference is principal.

(5) A *Cartier datum* is an open covering $\{U_i\}$ of X by non-empty opens $U_i \subset X$, together with elements $f_i \in K^*$ satisfying $f_i f_j^{-1} \in \mathcal{O}_X^*(U_i \cap U_j)$ for all i, j.

Lemma 3.3.11. *(1) For a Cartier divisor $D \in \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$, there exists an open cover $\{U_i\}$ (with $U_i \neq \emptyset$ for all i) of X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that for each i, j, we have $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. In other words, each Cartier divisor D defines a Cartier datum $\{(U_i, f_i)\}$.*

(2) Conversely, each Cartier datum $\{(U_i, f_i)\}$ defines a Cartier divisor D on X.

(3) Two Cartier data $\{(U_i, f_i)\}$ and $\{(V_j, g_j)\}$ define the same Cartier divisor if and only if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathcal{O}_X^*)$ for all i, j.

Proof. Exercise. $\square$

3.4 Lecture 23 : Cartier divisors, Weil divisors and sheaves

Lemma 3.4.1. *Let X be a noetherian normal integral scheme. Then there is a natural injective homomorphism*

$$\pi\colon \mathrm{CaDiv}(X) \longrightarrow \mathrm{Div}(X),$$

whose image consists of the Weil divisors D on X which are locally principal (in the sense of Corollary 3.3.8).

Proof. Let

$$\{(U_i, f_i)\}_{i\in I}$$

be a Cartier divisor on X. We define the associated Weil divisor $D = \sum n_Z Z$ as follows. Define a function

$$\varphi\colon \{\text{prime divisors } Z \subset X\} \longrightarrow I,$$

by choosing for each prime divisor $Z \subset X$, an element $i = \varphi(Z) \in I$ such that $U_i \cap Z \neq \emptyset$. We then put $n_Z \coloneqq \mathrm{ord}_{Z,X}(f_{\varphi(Z)})$.

This does not depend on the function φ. Indeed, if $U_j \cap Z \neq \emptyset$, then the generic point of Z is contained in U_i and in U_j, hence $U_i \cap U_j \neq \emptyset$, and $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*) \subset K$, so that $\mathrm{ord}_{Z,X}(f_i f_j^{-1}) = 0$, which implies that $\mathrm{ord}_{Z,X}(f_i) = \mathrm{ord}_{Z,X}(f_j)$.

Note that the sum $D = \sum n_Z Z = \sum \mathrm{ord}_{Z,X}(f_{\varphi(Z)}) Z$ is finite. Indeed, we can fix $i \in I$ such that $U_i \neq \emptyset$. Then the complement $W \coloneqq X - U_i$ is a closed subset of codimension at least one, which has finitely many irreducible components (since it is noetherian), hence there are finitely many prime divisors $Z \subset X$ which are contained in W; moreover, we can write

$$D = \sum_{Z\cap U_i \neq \emptyset} \mathrm{ord}_{Z\cap U_i, U_i}(f_i) Z + \sum_{Z \subset W} \mathrm{ord}_{Z,X}(f_{\varphi(Z)}) Z,$$

and the sum $\sum_{Z\cap U_i\neq\emptyset} \mathrm{ord}_{Z\cap U_i,U_i}(f_i) Z$ is finite because of Lemma 3.2.5.

This defines a group homomorphism, because if $\{(U_i, f_i)\}$ and $\{(V_j, g_j)\}$ are two Cartier divisors on X, then $\mathrm{ord}_{Z,X}(f_i \cdot g_j) = \mathrm{ord}_{Z,X}(f_i) + \mathrm{ord}_{Z,X}(g_j)$ for each prime divisor $Z \subset X$.

It remains to show that the image of π is the subgroup of locally principal divisors. It is clear that

$$D \coloneqq \pi\left(\{(U_i, f_i)\}_{i\in I}\right) = \sum_Z \mathrm{ord}_{Z,X}(f_{\varphi(Z)})$$

is locally principal for a Cartier divisor $\{(U_i, f_i)\}_{i\in I}$ on X, since $D|_{U_i}$ is principal for each $i \in I$. Conversely, if $D \in \mathrm{Div}(X)$ is locally principal, then there exists an open covering $X = \cup U_i$ such that $D|_{U_i} = \operatorname{div}(f_i) \in \operatorname{Div}(U_i)$ for some $f_i \in K^*$. Then $\{(U_i, f_i)\}$ gives a Cartier divisor on X. Indeed, we have

$$\operatorname{div}(f_i)|_{U_i \cap U_j} = D|_{U_i}|_{U_j} = D|_{U_j}|_{U_i} = \operatorname{div}(f_j)|_{U_i \cap U_j},$$

so that $f_i, f_j \in K^*$ define the same Weil divisor on $U_i \cap U_j$, which means that $\operatorname{div}(f_i f_j^{-1}) = 0$ as a Weil divisor on $U_i \cap U_j$, so that

$$\mathcal{O}_{U_i \cap U_j} = \mathcal{O}_{U_i \cap U_j}(\operatorname{div}(f_i f_j^{-1})) = f_i^{-1} f_j \cdot \mathcal{O}_{U_i \cap U_j} \subset \mathcal{K}|_{U_i \cap U_j},$$

see Lemmas 3.3.5 and 3.3.6. Therefore, $f_i^{-1} f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$.

To prove that π is injective, assume that $D = \sum \operatorname{ord}_{Z,X}(f_{\varphi(Z)})Z = 0$ for some Cartier divisor $\{(U_i, f_i)\}$ on X. Then $D|_{U_i} = 0$ for each $i \in I$, and $D|_{U_i} = \operatorname{div}(f_i) \in \operatorname{Div}(U_i)$. By the above argument, $f_i \in \Gamma(U_i, \mathcal{O}_X^*)$. Hence the Cartier divisor $\{(U_i, f_i)\}$ is trivial, that is, $\{(U_i, f_i)\} \in \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$ is the identity element. □

Definition 3.4.2. Let X be a noetherian integral normal scheme. For a Cartier divisor $D = \{(U_i, f_i)\}$ on X, we define a sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(D) := \mathcal{O}_X(\pi(D))$.

Lemma 3.4.3. *Let X be a normal integral noetherian scheme and let D and E be two Cartier divisors on X. Then the following hold:*

(1) $\mathcal{O}_X(D + E) \cong \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$;

(2) we have $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$ if and only if D and E are linearly equivalent.

Proof. As for item (1): choose an affine open cover $\{U_i\}$ of X such that $D|_{U_i} = \operatorname{div}(f_i)$ and $E|_{U_i} = \operatorname{div}(g_i)$. Let $i \in I$ and $A_i = \mathcal{O}_X(U_i)$. Since $\mathcal{O}_X(D)$, $\mathcal{O}_X(E)$ and $\mathcal{O}_X(D+E)$ are quasi-coherent, it suffices to show that the canonical map

$$\Gamma(U_i, \mathcal{O}_X(D)) \otimes_{A_i} \Gamma(U_i, \mathcal{O}_X(E)) \longrightarrow \Gamma(U_i, \mathcal{O}_X(D + E))$$

is an isomorphism for each $i \in I$. But as $D|_{U_i} = \operatorname{div}(f_i)$ and $E|_{U_i} = \operatorname{div}(g_i)$, this map can be identified with the map

$$\Gamma(U_i, \mathcal{O}_X(\operatorname{div}(f_i))) \otimes_{A_i} \Gamma(U_i, \mathcal{O}_X(\operatorname{div}(g_i))) \longrightarrow \Gamma(U_i, \mathcal{O}_X(\operatorname{div}(f_i g_i))$$

and as $\mathcal{O}_X(\operatorname{div}(f)) = f^{-1} \cdot \mathcal{O}_X$ for $f \in K^*$, this map corresponds to the map

$$f_i^{-1} A_i \otimes_{A_i} g_i^{-1} A_i \longrightarrow f_i^{-1} g_i^{-1} A_i. \tag{3.6}$$

The map (3.6) is an isomorphism, which proves (1).

To prove item (2), notice that, in view of item (1), it suffices to prove that $\mathcal{O}_X(D) \cong \mathcal{O}_X$ if and only if D is a principal Cartier divisor. For this, note that $\mathcal{O}_X(D) \cong \mathcal{O}_X$ if and only if $D = \operatorname{div}(f)$ is a principal Weil divisor (see Lemma 3.3.6), and moreover, for $D \in \operatorname{CaDiv}(X)$, have $\pi(D) = \operatorname{div}(f)$ for some $f \in K^*$ if and only if D is in the image of $K^* \to \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$, which is to say, D is a principal Cartier divisor. □

Proposition 3.4.4. *Let X be a noetherian normal integral scheme. Then the association $D \mapsto \mathcal{O}_X(D)$ defines a natural homomorphism*

$$\rho\colon \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X),$$

where, as usual, $\mathrm{Pic}(X)$ *denotes the group of isomorphism classes of invertible sheaves.*

Proof. This follows from Corollary 3.3.8 together with Lemmas 3.4.1 and 3.4.3. □

Proposition 3.4.5. *Let X be a noetherian normal integral scheme. The map*

$$\rho\colon \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X), \qquad D \mapsto \mathcal{O}_X(D)$$

is an isomorphism.

Proof. Injectivity: If D and E are Cartier divisors with $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$, then D is linearly equivalent to E (see Lemma 3.4.3), so that ρ is injective.

Surjectivity: Let $\mathcal{L}$ be a line bundle on X. Let $V \subset X$ be a non-empty open subset of X such that there exists a non-zero section $f \in \Gamma(V, \mathcal{L})$. Let $\{U_i\}$ be an open cover of X by non-empty affine opens $U_i \subset X$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Then f induces elements $f_i \in \Gamma(U_i \cap V, \mathcal{O}_X) \subset K$; in particular, we get $f_i \in K$ for each $i \in I$. We remark that $D := \{U_i, f_i\}$ is a Cartier divisor, and that $\mathcal{O}_X(D) \cong \mathcal{L}$ (exercise). □

Lemma 3.4.6. *Let $X = \mathrm{Spec}\ A$ where A is a noetherian normal domain. Let $D \in \mathrm{Div}(A)$, write $D = \sum_{\mathfrak{q}} n_{\mathfrak{q}} V(\mathfrak{q})$, where $\mathfrak{q}$ ranges over the primes in A of height one. Consider the sheaf $\mathcal{O}_X(D)$ on X. Let $\mathfrak{p} \in X$. Define a Weil divisor $D_{\mathfrak{p}}$ on $\mathrm{Spec}\ A_{\mathfrak{p}}$ as*

$$D_{\mathfrak{p}} := \sum_{\mathfrak{q}} n_{\varphi(q)} V(\mathfrak{q}), \quad (\textit{with} \quad \varphi\colon \mathrm{Spec}\ A_{\mathfrak{p}} \to \mathrm{Spec}\ A \quad \textit{the canonical morphism}),$$

where $\mathfrak{q}$ ranges over the prime ideals of $A_{\mathfrak{p}}$ of height one. Then $\mathcal{O}_X(D)_{\mathfrak{p}} = \mathcal{O}_{\mathrm{Spec}\ A_{\mathfrak{p}}}(D_{\mathfrak{p}})$.

Proof. Exercise. □

Lemma 3.4.7. *Let A be a noetherian ring. Let M be a finitely generated A-module. Let $X = \mathrm{Spec}\ A$, $\mathfrak{p} \in X$, and $n \in \mathbb{Z}_{\geq 1}$. The coherent sheaf $\mathcal{F} = \widetilde{M}$ is locally free (of rank n) around $\mathfrak{p} \in X$ if and only if the stalk $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module (of rank n).*

Proof. Exercise. □

Proposition 3.4.8. *Let X be a normal integral noetherian scheme. Suppose that $\mathcal{O}_{X,x}$ is a UFD for each $x \in X$. The map $\pi\colon \mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ is an isomorphism.*

Proof. Let $D \in \mathrm{Div}(X)$. We need to show that D is locally principal (see Lemma 3.4.1), or equivalently, that $\mathcal{O}_X(D)$ is locally free of rank one (see Lemma 3.3.8). Equivalently (see Lemma 3.4.7), we need to show that $\mathcal{O}_X(D)_x$ is a free $\mathcal{O}_{X,x}$-module for each $x \in X$. Thus we may assume that $X = \mathrm{Spec}\ A$ is affine, and need to show that $\mathcal{O}_X(D)_x$ is a free $\mathcal{O}_{X,x}$-module, for any $x \in X$ corresponding to a prime $\mathfrak{p} \subset A$. To then prove that $\mathcal{O}_X(D)_x \cong \mathcal{O}_{X,x}$, it suffices, in view of Lemma 3.4.6, to prove that $\mathcal{O}_X(D) \cong \mathcal{O}_X$ for any $D \in \mathrm{Div}(\mathrm{Spec}\ A)$ in case A is a local noetherian domain and a UFD. But in this case, we have $\mathrm{Cl}(\mathrm{Spec}\ A) = 0$ by Proposition 3.2.17, so that D is principal, and hence $\mathcal{O}_X(D) \cong \mathcal{O}_X$ (see Lemma 3.3.6) as desired. □

Theorem 3.4.9. *Let X be a smooth variety over a field k. Then for each $x \in X$, we have $\dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim(\mathcal{O}_{X,x})$, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ denotes the maximal ideal of $\mathcal{O}_{X,x}$. In particular, the local ring $\mathcal{O}_{X,x}$ is a UFD for each $x \in X$.*

Proof. We do not prove this here. □

Corollary 3.4.10. *Let X be a smooth variety over a field. Then the natural maps*

$$\pi\colon \mathrm{CaCl}(X) \longrightarrow \mathrm{Cl}(X) \qquad \text{and} \qquad \rho\colon \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X)$$

are isomorphisms. □

Corollary 3.4.11. *Let k be a field. Then $\mathrm{Pic}(\mathbb{A}_k^n) = 0$. In particular, for the k-algebra $R = k[x_1, \dots, x_n]$, we have that any R-module M such that $M_\mathfrak{p} \cong R_\mathfrak{p}$ for each prime ideal $\mathfrak{p} \subset R$ is in fact isomorphic to R.*

Proof. Indeed, we saw that $\mathrm{Cl}(\mathbb{A}_k^n) = 0$, see Corollary 3.2.18. Thus $\mathrm{Pic}(\mathbb{A}_k^n) = 0$ by Corollary 3.4.10. The other statement follows then from Lemma 3.4.7. □

3.4.1 Restricting divisors to an open subscheme

Theorem 3.4.12. *Let X be a noetherian normal integral scheme and let $U \subset X$ be a non-empty open subscheme. Let $W = X - U$. Let $\mathcal{Z}_1, \dots, \mathcal{Z}_r \subset W$ be the irreducible components of W of codimension one in X. Then the natural sequence*

$$\bigoplus_{i=1}^{r} \mathbb{Z} \cdot [\mathcal{Z}_i] \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0$$

is exact.

Proof. Right exactness follows from the fact that if $Z \subset U$ is a prime divisor on U, then its closure $\overline{Z} \subset X$ is a prime divisor on X, and $\overline{Z}|_U = \overline{Z} \cap U = Z$. It is also clear that for any $i \in \{1, \dots, r\}$, we have $\mathcal{Z}_i|_U = 0$ as Weil divisors on U. Conversely, let $Z \subset X$ be a prime divisor such that $Z|_U = \mathrm{div}(f)$ for some $f \in K^*$. Then $D := Z - \mathrm{div}(f) \in \mathrm{Div}(X)$ satisfies $D|_U = 0$ as Weil divisors on U. Thus D is supported on W, hence $D = \sum_{i=1}^r n_i \mathcal{Z}_i$ for some $n_i \in \mathbb{Z}$, and the theorem follows. □

Corollary 3.4.13. *Let X be a noetherian normal integral scheme. Let $Y \subset X$ be an integral closed subscheme of codimension at least two. Then the natural map*

$$\mathrm{Cl}(X) \to \mathrm{Cl}(X - Y)$$

is an isomorphism. □

Example 3.4.14. Let k be a field. The map $\mathrm{Cl}(\mathbb{A}_k^2) \to \mathrm{Cl}(\mathbb{A}_k^2 - \{0\})$ is an isomorphism. Thus $\mathrm{Cl}(\mathbb{A}_k^2 - \{0\}) = 0$ in view of Corollary 3.2.18.

Example 3.4.15. Let k be a field and let $P \in \mathbb{P}^1_k$ be the image of a k-point $\operatorname{Spec} k \to \mathbb{P}^1_k$ (cf. Example 3.1.28). Let $U = \mathbb{P}^1_k - \{P\}$. Then $U \cong \mathbb{A}^1_k$. We get an exact sequence

$$\mathbb{Z} \cdot [P] \longrightarrow \mathrm{Cl}(\mathbb{P}^1_k) \longrightarrow \mathrm{Cl}(\mathbb{A}^1_k) = 0.$$

This gives a surjection $\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^1_k)$ sending 1 to the class of P in $\mathrm{Cl}(\mathbb{P}^1_k)$. This map is injective, for if $[nP] = 0 \in \mathrm{Cl}(\mathbb{P}^1_k)$, then $nP = \operatorname{div}(f)$ for some $f \in k(\mathbb{P}^1_k)$, and we have $nP|_U = 0$ so that $\operatorname{div}(f)|_U = 0$, which implies that $f \in k(\mathbb{P}^1_k) = k(t)$ has neither zeros nor poles on $U \cong \mathbb{A}^1_k$ so that $f \in \Gamma(\mathbb{A}^1_k, \mathcal{O}^*_{\mathbb{A}^1_k}) = k^*$. Hence f is constant, so $n = 0$.

We conclude that $\mathrm{Cl}(\mathbb{P}^1_k) = \mathbb{Z} \cdot [P]$ (compare Exercise 3.3.1). Under the isomorphism $\mathrm{Cl}(\mathbb{P}^1_k) = \mathrm{Pic}(\mathbb{P}^1_k)$, the generator P is sent to $\mathcal{O}_{\mathbb{P}^1_k}(1)$, hence this also shows that $\mathrm{Pic}(\mathbb{P}^1_k) = \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^1_k}(1)$ (compare Exercise 2.3.11).

Chapter 4

Differentials

In this chapter, we introduce Kähler differentials, which allow us to the sheaf of Kähler differentials for an algebraic variety. Differentials appear in many areas of mathematics, including multivariable analysis, differential geometry and complex geometry. In algebraic geometry, they are introduced algebraically, and referred to as Kähler differentials. For a smooth variety, the sheaf of Kähler differentials is locally free of rank equal to the dimension of the variety, and forms the algebraic analogue of the cotangent bundle of a smooth manifold. Namely, if the variety is defined over a perfect field, then for each point of the variety, the fibre of this sheaf above this point is canonically isomorphic to the dual of the Zariski tangent space of the variety at that point.

4.1 Lecture 24 : Kähler differentials

Definition 4.1.1. Let $A \to B$ be a morphism of rings. Let M be a B-module. Then an A-*derivation* from B with values in M is an A-linear map $D\colon B \to M$ such that, for all $b_1, b_2 \in B$, we have

$$D(b_1 b_2) = b_1 D(b_2) + b_2 D(b_1).$$

Note that, in particular, $D(a) = 0$ for every $a \in A$ (indeed, this follows from the string of equalities $D(a) = a \cdot D(1 \cdot 1) = 2a \cdot D(1) = 2 \cdot D(a)$).

Let $\mathrm{Der}_A(B, M)$ be the set of A-derivations of B into M. Note that $\mathrm{Der}_A(B, M)$ has a natural B-module structure.

Example 4.1.2. Let $t_1, \ldots, t_n$ be variables and let $B = M = k[t_1, \ldots, t_n]$. Then for each $i \in \{1, \ldots, n\}$, we have that $\partial/\partial t_i\colon B \to B$ is an A-derivation.

Note that, for a fixed morphism of rings $A \to B$, the association

$$M \mapsto \mathrm{Der}_A(B, M)$$

defines a functor $\mathrm{Der}_A(B, -)$ from Mod_B to itself.

Proposition 4.1.3. *Let $A \to B$ be a morphism of rings. The covariant functor*

$$\mathrm{Der}_A(B, -)\colon \mathsf{Mod}_B \longrightarrow \mathsf{Mod}_B$$

is representable. In other words, there exists a B-module $\Omega_{B/A}$ and a derivation

$$d_B\colon B \longrightarrow \Omega_{B/A}$$

such that for any B-module M and any A-derivation $D\colon B \to M$ there exists a unique B-module morphism $\alpha\colon \Omega_{B/A} \to M$ such that $\alpha \circ d_B = D$.

Proof. Define G as the free B-module on the set underlying B, and for $b \in B$, let $db \in G$ be the canonically attached element. Thus $G = \bigoplus_{b \in B} B \cdot db$. Let $H \subset G$ be the submodule generated over B by elements of the form

$$d(b + b') - db - db' \quad \text{and} \quad d(bb') - bdb' - b'db \quad \text{and} \quad da,$$

for $a \in A, b, b' \in B$. Define $\Omega_{B/A} = G/H$, and the map $d_B\colon B \to \Omega_{B/A}$ as the map with $d_B(b) = db$. This is a group homomorphism (by the first constraint), it satisfies $d(bb') = bd(b') + b'd(b)$ for $b, b' \in B$ (by the second constraint), and it is A-linear (since $d_B(A) = 0$ and $d(bb') = bd(b') + b'd(b)$ for $b, b' \in B$).

Let $D\colon B \to M$ be an A-derivation into a B-module M. Define a B-morphism $\alpha\colon \Omega_{B/A} \to M$ by putting $\alpha(db) = D(b)$ for $b \in B$ and extending linearly. Then α is the unique B-module map $\Omega_{B/A} \to M$ such that $\alpha \circ d = D$. $\square$

Example 4.1.4. Let $A \to A'$ be a morphism of rings, let $A' \to B$ be a morphism of rings, and let M be a B-module. We get an inclusion $\mathrm{Der}_{A'}(B, M) \subset \mathrm{Der}_A(B, M)$. In particular, $\mathrm{Der}_{A'}(B, \Omega_{B/A'}) \subset \mathrm{Der}_A(B, \Omega_{B/A'})$. Thus, by the universal property, the A-derivation $d_{B/A'}\colon B \to \Omega_{B/A'}$ factors as

$$B \longrightarrow \Omega_{B/A} \longrightarrow \Omega_{B/A'}.$$

Proposition 4.1.5. *Let A be a ring, $B = A[t_1, \ldots, t_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $dt_1, \ldots, dt_n$, and $d_B\colon B \to \Omega_{B/A} = \bigoplus_i Bdt_i$ is defined as*

$$d_B(f) = \sum (\partial f/\partial t_i) dt_i, \qquad f \in B = A[t_1, \ldots, t_n].$$

Proof. Consider the map $d\colon B \to \bigoplus_i Bdt_i$ with $d(f) = \sum(\partial f/\partial t_i)dt_i$ for $f \in B$. Let M be any B-module, and $D\colon B \to M$ an A-derivation. There is a unique morphism of B-modules $\bigoplus_i Bdt_i \to M$ making the obvious triangle commute (exercise). $\square$

Proposition 4.1.6. *Let $A \to B$ be a morphism of rings. Let $C = B/I$ for some ideal $I \subset B$. Let $\alpha\colon B \to C$ be the quotient map. Note that I/I^2 is a C-module in a canonical way. The sequence of C-modules*

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \xrightarrow{f} \Omega_{C/A} \longrightarrow 0$$

is exact, where $\delta(\bar{x}) = dx \otimes 1$ for $x \in I$ with image $\bar{x} \in I/I^2$, and where $f(db \otimes c) = c \cdot db$.

Proof. It suffices to show that for each C-module N, the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_C(\Omega_{C/A}, N) \arrow[r, "f^*"] \arrow[d, equal] & \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, N) \arrow[r, "\delta^*"] \arrow[d, equal] & \mathrm{Hom}_C(I/I^2, N) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Der}_A(C, N) \arrow[r] & \mathrm{Der}_A(B, N) \arrow[r] & \mathrm{Hom}_C(I \otimes_B C, N) \arrow[d, equal] \\
& & & \mathrm{Hom}_B(I, N)
\end{tikzcd}$$

is exact. Since $\alpha\colon B \to C$ is surjective, the map f is surjective, too (verify this). Thus, the map $f^*\colon \mathrm{Der}_A(C, N) \longrightarrow \mathrm{Der}_A(B, N)$ is injective. The map

$$\delta^*\colon \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Hom}_B(I, N)$$

associates to an A-derivation $D\colon B \to N$ its restriction to I, which is an A-linear map $D|_I\colon I \to N$, which is in fact B-linear, since for $b \in B$ and $a \in I$, we have $D(b \cdot a) = aD(b) + bD(a) = bD(a) \in N$. If $D|_I = 0$ then D factors as

$$B \xrightarrow{\alpha} C \xrightarrow{D'} N,$$

and the induced map $D'\colon C \to N$ is an A-derivation from C into N. □

Proposition 4.1.7. *Let A be a ring and let B be a finitely generated A-algebra. Then $\Omega_{B/A}$ is finitely generated over B.*

Proof. Write $B = A[t_1, \dots, t_n]/I$ for some ideal $I \subset A[t_1, \dots, t_n]$. Then we get an exact sequence

$$I/I^2 \longrightarrow \Omega_{A[t_1,\dots,t_n]/A} \otimes_A B \longrightarrow \Omega_{B/A} \longrightarrow 0.$$

As $\Omega_{A[t_1,\dots,t_n]/A} = \bigoplus_{i=1}^n A dt_i$ (see Example 4.1.2), we are done. □

4.1.1 Kähler differentials on schemes

Definition 4.1.8. Let $f\colon X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then an $f^{-1}(\mathcal{O}_S)$-linear morphism $D\colon \mathcal{O}_X \to \mathcal{F}$ is called an *$\mathcal{O}_S$-derivation* if for all affine open subset $V \subset S$ and $U \subset X$ with $f(U) \subset V$, the map

$$D|_U\colon \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$$

is an $\mathcal{O}_S(V)$-derivation (with respect to the natural ring morphism $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$).

For a morphism of schemes $f\colon X \to S$, and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, let $\mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{F})$ denote the set of $\mathcal{O}_S$-derivations $\mathcal{O}_X \to \mathcal{F}$, which is naturally an $\mathcal{O}_X(X)$-module. Remark that the association

$$\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{F})$$

defines a functor

$$\mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, -)\colon \mathsf{QCoh}(X) \longrightarrow \mathsf{Mod}_{\mathcal{O}_X(X)} \tag{4.1}$$

from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(X)$-modules.

Theorem 4.1.9. *Let $f\colon X \to S$ be a morphism of schemes.*

(1) The functor (4.1) is representable. In other words, there exists a quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/S}$ together with an $\mathcal{O}_S$-derivation

$$d_X\colon \mathcal{O}_X \to \Omega_{X/S}$$

that satisfies the following universal property: for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_S$-derivation $D\colon \mathcal{O}_X \to \mathcal{F}$, there exists a unique morphism of $\mathcal{O}_X$-modules $\alpha\colon \Omega_{X/S} \to \mathcal{F}$ such that $D = \alpha \circ d_X$.

(2) The sheaf $\Omega_{X/S}$ is a quasi-coherent sheaf on X that has the property that for each affine open $V = \operatorname{Spec} A \subset S$ and each affine open $U = \operatorname{Spec} B \subset f^{-1}(V) \subset X$, we have a canonical isomorphism of $\mathcal{O}_U$-modules

$$\Omega_{X/S}|_U \cong \widetilde{\Omega_{B/A}}.$$

(3) For each $x \in X$, we have a canonical isomorphism $(\Omega_{X/S})_x \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}$.

(4) If f is of finite type, then $\Omega_{X/S}$ is coherent.

Proof. Items (1), (2) and (3): either prove this by adapting the proof of the affine case, or use the result from the affine case to define $\Omega_{X/S}|_U$ for affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$, and then glue. For item (4), see Proposition 4.1.7. □

4.1.2 Euler sequence

Theorem 4.1.10. *Let A be a ring. There is a natural exact sequence*

$$0 \longrightarrow \Omega_{\mathbb{P}^n_A/A} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow 0.$$

Proof. We do not prove this here. See e.g. [OE15, Theorem 19.24]. □

4.2 Lecture 25 : Regular and smooth schemes

4.2.1 Regular local rings

Lemma 4.2.1. *Let B be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $e \in \mathbb{Z}_{\geq 0}$. Let $K = B/\mathfrak{m}$. Then $\mathfrak{m}$ can be generated by e elements if and only if $\dim_K(\mathfrak{m}/\mathfrak{m}^2) \leq e$.*

Proof. Assume $\mathfrak{m} = (x_1, \ldots, x_e)$ can be generated by e elements. Then $\dim_K(\mathfrak{m}/\mathfrak{m}^2) \leq e$. Conversely, let $x_1, \ldots, x_e \in \mathfrak{m}$ be such that they generate $\mathfrak{m}/\mathfrak{m}^2$. We get an inclusion $I := (x_1, \ldots, x_e) \subset \mathfrak{m}$. Consider the ring $\bar{R} := R/I$. Let $\bar{\mathfrak{m}} \subset \bar{R}$ be the image of $\mathfrak{m}$. We get $\bar{\mathfrak{m}} \equiv 0 \bmod \bar{R}/\bar{\mathfrak{m}}^2$. Hence $\bar{\mathfrak{m}} = \bar{\mathfrak{m}}^2 \subset \bar{R}$. Thus $\bar{\mathfrak{m}} = 0$ by Nakayama's lemma. □

Lemma 4.2.2. *Let B be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $K = B/\mathfrak{m}$. Then $\dim(B) \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2)$.*

Proof. Note that $\mathrm{ht}(\mathfrak{m}) = \dim(B)$: the height of $\mathfrak{m}$ equals the dimension of B. Now assume $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = e$. By Lemma 4.2.1, we see that $\mathfrak{m}$ can be generated by e elements. Thus $\mathfrak{m} = (x_1, \dots, x_e)$ for some $x_i \in B$. This implies $\mathrm{ht}(\mathfrak{m}) \leq e$ (verify this). Therefore, we get $\dim(B) \leq e = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$ as desired. □

Definition 4.2.3. Let B be a noetherian local ring with maximal ideal $\mathfrak{m}$. Then B is called *regular* if there exist $n := \dim(B)$ elements $x_1, \dots, x_n$ with $\mathfrak{m} = (x_1, \dots, x_n)$.

Lemma 4.2.4. *B is regular if and only if* $\dim(B) = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$, *where* $K = B/\mathfrak{m}$.

Proof. Let $n = \dim(B)$. By Lemma 4.2.2, $n \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2)$. By Lemma 4.2.1, we have that $\mathfrak{m}$ can be generated by n elements if and only if $\dim_K(\mathfrak{m}/\mathfrak{m}^2) \leq n$. Combining this, we see that $\mathfrak{m}$ can be generated by n elements if and only if $n \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2) \leq n$. □

4.2.2 Regular schemes

Definition 4.2.5. Let X be a noetherian scheme and let $x \in X$. We say that X is *regular at x* if the noetherian local ring $\mathcal{O}_{X,x}$ is regular. We say that X is *regular* if X is regular at all of its points.

Definition 4.2.6. Let X be a scheme and let $x \in X$. The *Zariski tangent space* of X at x, denoted by T_xX, is the $k(x)$-vector space

$$T_xX := (\mathfrak{m}/\mathfrak{m}^2)^\vee = \mathrm{Hom}_{k(x)}(\mathfrak{m}/\mathfrak{m}^2, k(x))$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,x}$ and $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}$ is the residue field of x.

Lemma 4.2.7. *Let X be a noetherian scheme and let $x \in X$.*

(1) The scheme X is regular at x if and only if $\dim(\mathcal{O}_{X,x}) = \dim_{k(x)}(T_xX)$.

(2) If X is a variety over a field k, and if $x \in X$ is a closed point (cf. Proposition 3.1.11), then X is regular at x if and only if $\dim(X) = \dim_{k(x)}(T_xX)$.

Proof. Item (1) follows from Lemma 4.2.4 and the definitions. Item (2) follows from item (1) together with the equality

$$\dim(X) = \mathrm{codim}(\overline{\{x\}}, X) = \dim(\mathcal{O}_{X,x}),$$

see Proposition 3.1.25. □

Exercise 4.2.8. Let X be a regular noetherian scheme. Show that X is reduced, that is, that $\mathcal{O}_{X,x}$ is reduced for each $x \in X$.

4.2.3 Regular schemes and Kähler differentials

Proposition 4.2.9. *Let $(B, \mathfrak{m})$ be a local k-algebra with residue field $K = B/\mathfrak{m} \supset k$. Suppose that $k \subset K$ is finite and separable. Then the map*

$$\delta\colon \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \otimes_B K$$

from the conormal sequence (4.1.6) is an isomorphism.

Proof. Note that the conormal sequence (4.1.6) reads

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \otimes_B K \longrightarrow \Omega_{K/k} \longrightarrow 0.$$

We claim that $\Omega_{K/k} = 0$. Indeed, K is finite and separable over k, so that by the primitive element theorem, we have

$$K = k(\alpha) = k[x]/(f)$$

for some $f \in f[x]$ with $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Let M be a K-vector space and let $d\colon K \to M$ be a k-derivation. Then

$$0 = d(0) = d(f(\alpha)) = f'(\alpha) \cdot d(\alpha).$$

Since $f'(\alpha) \in K^*$, it follows that $d(\alpha) = 0$. Now K is generated as a k-vector space by the powers α^i for $i \in \mathbb{Z}_{\geq 0}$, and we have $d(\alpha^i) = i \cdot \alpha^{i-1} \cdot d(\alpha) = 0$ for each $i \geq 1$. Thus $d = 0$. By the universal property of $d_K\colon K \to \Omega_{K/k}$, we conclude that $\Omega_{K/k} = 0$.

It remains to verify that $\delta\colon \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B K$ is injective. We leave this as an exercise for the reader. $\square$

Corollary 4.2.10. *Let B be an algebra satisfying the assumptions in Proposition 4.2.9. Assume in addition that B is noetherian. Then B is a regular local ring if and only if*

$$\dim(B) = \dim_k \left(\Omega_{B/k} \otimes_B K\right).$$

Proof. By Proposition 4.2.9, we see that $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = \dim_K(\Omega_{B/k} \otimes_k K)$. $\square$

Recall that a field k is called *perfect* if every algebraic extension k'/k of k is separable. Examples include algebraically closed fields, finite fields, and fields of characteristic zero. For a non-example: the field $\mathbb{F}_p(t)$ is not perfect. Namely, the extension $\mathbb{F}_p(t) \subset \mathbb{F}_p(t^{1/p})$ is not separable.

For varieties over a perfect field X, Kähler differentials are closely related to tangent vectors at closed points (i.e., elements of $T_x X$ for $x \in X$ closed).

Corollary 4.2.11. *Let X be a variety over a perfect field k. Let $x \in X$ be a closed point. Then there is a canonical isomorphism of $k(x)$-vector spaces*

$$(T_x X)^\vee \xrightarrow{\sim} \Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} k(x).$$

Proof. This is clear from Proposition 4.2.9, as the field extension $k \subset k(x)$ is finite (see Proposition 3.1.11) hence separable since k is perfect. $\square$

This justifies why the sheaf $\Omega_{X/k}$ is often called the *cotangent bundle*. However, note that $\Omega_{X/k}$ is not a vector bundle in general, that is, this $\mathcal{O}_X$-module is not always locally free of finite rank. If X is smooth, then this turns out to be the case, as we will show next. Conversely, if k is perfect and $\Omega_{X/k}$ is locally free, then X is smooth.

4.2.4 Smooth schemes, regular schemes and Kähler differentials

Definition 4.2.12. Let k be a field. Let $X \subset \mathbb{A}^n_k$ be a closed subscheme of $\mathbb{A}^n_k$, cut out by polynomials $f_1, \dots, f_r \in k[t_1, \dots, t_n]$. Let $x \in X$ be a closed point, cf. Proposition 3.1.11. Remark that, in view of Lemma 2.4.2, we can associate to $x \in X$ an element

$$x \in \mathrm{Hom}_k(\mathrm{Spec}\ k(x), X) = X(k(x)) \subset \mathbb{A}^n(k(x)) = (k(x))^n\,,$$

We define the *Jacobian matrix* of $X \subset \mathbb{A}^n_k$ at the point x as follows:

$$J_x := \left(\frac{\partial f_i}{\partial t_j}(x)\right)_{1 \le i \le r, 1 \le j \le n} \in \mathrm{M}_{r \times n}(k(x)). \tag{4.2}$$

Lemma 4.2.13. *Let k be a field. Let $X \subset \mathbb{A}^n_k$ be a closed subscheme of $\mathbb{A}^n_k$, cut out by polynomials $f_1, \dots, f_r \in k[t_1, \dots, t_n]$. Let $x \in X$ be a closed point. Then*

$$\dim_{k(x)}(T_x X) \le n - \mathrm{rank}(J_x), \tag{4.3}$$

with $J_x \in \mathrm{M}_{r \times n}(k(x))$ as in (4.2). If $k = \bar{k}$, then (4.3) is an equality.

Proof. Exercise. □

Definition 4.2.14. Let X be a scheme of finite type over a field k. Let $x \in X$ be a closed point (cf. Proposition 3.1.11).

(1) Assume $k = \bar{k}$. We say that X is *smooth at* x if there exists an affine open neighbourhood U of x in X with $U \cong \mathrm{Spec}\ A \subset \mathbb{A}^n_k$ for a finitely generated k-algebra $A = k[t_1, \dots, t_n]/(f_1, \dots, f_r)$, such that the rank $\mathrm{rk}(J_x)$ of the matrix (4.2) satisfies the equality

$$\mathrm{rk}(J_x) = n - \dim(\mathcal{O}_{X,x}).$$

Remark that if X is irreducible, then $\dim(\mathcal{O}_{X,x}) = \dim(X)$ since x is closed, see Proposition 3.1.25.

(2) In general, we say that X is *smooth at* x if for any closed point $x' \in X_{\bar{k}} = X \times_k \bar{k}$ lying over x, the scheme $X_{\bar{k}}$ is smooth at x'. We say that the scheme X is *smooth over* k if X is smooth at x for any $x \in X$.

Proposition 4.2.15. *Let X be a scheme of finite type over a field k. Let $x \in X$ be a closed point. Then the following assertions are equivalent:*

(1) The scheme X is smooth at x.

(2) For any closed point $x' \in X_{\bar{k}}$ lying over $x \in X$, the scheme $X_{\bar{k}}$ is smooth at x'.

(3) For any closed point $x' \in X_{\bar{k}}$ lying over $x \in X$, the scheme $X_{\bar{k}}$ is regular at x'.

(4) For any affine open neighbourhood U of x in X with $U \cong \mathrm{Spec}\ A \subset \mathbb{A}^n_k$ for a finitely generated k-algebra $A = k[t_1, \dots, t_n]/(f_1, \dots, f_r)$, the rank $\mathrm{rk}(J_x)$ of the matrix (4.2) satisfies the equality $\mathrm{rk}(J_x) = n - \dim(\mathcal{O}_{X,x})$.

(5) *There exists an affine open neighbourhood U of x in X with $U \cong \operatorname{Spec} A \subset \mathbb{A}^n_k$ for a finitely generated k-algebra $A = k[t_1, \dots, t_n]/(f_1, \dots, f_r)$, such that the rank* $\operatorname{rk}(J_x)$ *of the matrix* (4.2) *satisfies the equality* $\operatorname{rk}(J_x) = n - \dim(\mathcal{O}_{X,x})$.

Proof. Assume (1). Then (2) holds by definition. Assume (2) and let $x' \in X_{\bar{k}}$ be a closed point lying over $x \in X$. We claim that $X_{\bar{k}}$ is regular at x'. Since $X_{\bar{k}}$ is smooth at x', there exists an affine open neighbourhood V of x' in $X_{\bar{k}}$ with $V \cong \operatorname{Spec} A \subset \mathbb{A}^n_{\bar{k}}$ for a finitely generated $\bar{k}$-algebra $A = \bar{k}[t_1, \dots, t_n]/(f_1, \dots, f_r)$, such that the rank $\operatorname{rk}(I_{x'})$ of the matrix $I_{x'} = (\partial f_i/\partial t_j(x'))$ satisfies the equality

$$\operatorname{rk}(I_{x'}) = n - \dim(\mathcal{O}_{X_{\bar{k}},x'}).$$

By Lemma 4.2.13, we have $\dim_{\bar{k}}(T_{x'}X_{\bar{k}}) = n - \operatorname{rk}(I_{x'})$. It follows that $\dim(\mathcal{O}_{X_{\bar{k}},x'}) = \dim_{\bar{k}}(T_{x'}X_{\bar{k}})$. Thus, $X_{\bar{k}}$ is regular at $x' \in X_{\bar{k}}$, proving (3). We claim that (4) also holds. Namely, let U be any affine open neigbourhood of $x \in X$ with $U \cong \operatorname{Spec} B \subset \mathbb{A}^n_k$ for a finitely generated k-algebra $B = k[t_1, \dots, t_n]/(g_1, \dots, g_s)$. Let $J_x = (\partial g_i/\partial t_j(x))_{ij}$ and $J_{x'} = (\partial g_i/\partial t_j(x'))_{ij}$. Then $\operatorname{rk}(J_{x'}) = \operatorname{rk}(J_x)$ and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X_{\bar{k}},x'})$. By Lemma 4.2.13, we have $\dim_{\bar{k}}(T_{x'}X_{\bar{k}}) = n - \operatorname{rk}(J_{x'})$. Therefore:

$$\operatorname{rk}(J_x) = \operatorname{rk}(J_{x'}) = n - \dim_{\bar{k}}(T_{x'}X_{\bar{k}}) = n - \dim(\mathcal{O}_{X_{\bar{k}},x'}) = n - \dim(\mathcal{O}_{X,x}).$$

This proves (4) as desired.

Clearly, (4) proves (5). Finally, assume (5). Let $x' \in X_{\bar{k}}$ be a closed point lying over $x \in X$. Let $U \subset X$ as in (5). We have $U_{\bar{k}} \cong \operatorname{Spec}(A \otimes_k \bar{k}) \subset \mathbb{A}^n_{\bar{k}}$, and $\operatorname{rk}(J_{x'}) = \operatorname{rk}(J_x) = n - \dim(\mathcal{O}_{X,x}) = n - \dim(\mathcal{O}_{X_{\bar{k}},x'})$, proving (1). We are done. □

Lemma 4.2.16. *Let X be an irreducible scheme of finite type over a field k. Let $x \in X$ be a closed point. Assume that X is smooth at x. Then there exists an open neighbourhood $x \in U \subset X$ of x in X such that the scheme U is smooth over k.*

Proof. We may assume $X = \operatorname{Spec} k[t_1, \dots, t_n]/(f_1, \dots, f_r)$ with the Jacobian matrix $(\partial f_i/\partial t_j(x))$ of rank $m := n - \dim(X)$ at $x \in X$. Thus, there exists a $m \times m$-minor of the matrix $(\partial f_i/\partial t_j)$ which does not vanish at $x \in X$. Hence this minor does not vanish in an open neighbourhood U of x in X. This scheme U is smooth over k. □

Exercise 4.2.17. Let X be a scheme of finite type over a field k. Let X^{cl} be the set of closed points of X. Show that X^{cl} is dense in X.

Exercise 4.2.18. Let $\mathcal{F}$ be a coherent sheaf on a noetherian scheme X. Define a function $\phi\colon X \to \mathbb{Z}$ as $\phi(x) = \dim_{k(x)}(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x))$.

(1) Let $n \in \mathbb{Z}_{\geq 1}$. Using Nakayama's lemma, show that the set $\{x \in X \mid \phi(x) \leq n\}$ is open in X.

(2) Deduce that if X is irreducible with generic point η, then we have $\phi(x) \geq \phi(\eta)$ for all $x \in X$.

(3) Let us suppose that ϕ is constant of value $n \geq 1$ on X, and that X is reduced. Show that $\mathcal{F}$ is locally free of rank n.

(4) Assume that X is an irreducible scheme of finite type over a field k. Define $U := \{x \in X \mid \phi(x) \leq \phi(\eta)\}$, where $\eta \in X$ is the generic point. Let $X^{\mathrm{cl}} \subset X$ be the set of closed points of X. Show that $U \cap X^{\mathrm{cl}} \neq \emptyset$. Conclude that there exists a closed point $x \in X$ with $\phi(x) = \phi(\eta)$.

(5) Assume X is an integral scheme of finite type over a field k. Let $n \in \mathbb{Z}_{\geq 1}$. Suppose $\phi(x) = n$ for every closed $x \in X$. Show that $\mathcal{F}$ is locally free of rank n.

Lemma 4.2.19. *Let X be a scheme of finite type over a field k. Let $k' \supset k$ be an algebraic field extension, and let $X' = X \times_k k'$. Let $p\colon X' \to X$ be the natural map. Let $x' \in X'$ and $x \in X$ be closed points with $x = p(x')$.*

(1) There is a natural isomorphism $p^*(\Omega_{X/k}) \cong \Omega_{X'/k'}$.

(2) In particular, there is a natural isomorphism $\Omega_{X'/k',x'} \cong \Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ and hence $\Omega_{X'/k',x'} \otimes_{\mathcal{O}_{X',x'}} k(x') \cong \big(\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} k(x)\big) \otimes_{k(x)} k(x')$.

(3) Consequently, if k is perfect, we have a natural isomorphism of $k(x')$-vector spaces $T_xX \otimes_{k(x)} k(x') \cong T_{x'}X'$.

Proof. Exercise. □

Non-Example 4.2.20. Let t be a variable, let p be a prime number, and let $k = \mathbb{F}_p(t)$. We consider the inseparable field extension $k \subset k(\alpha)$ where $\alpha^p = t$. Let $C \subset \mathbb{A}^2_k$ be the curve defined by the equation $x^p + y^p = t$. Then $C' := C \times_k k(\alpha)$ is given by the equation

$$(x+y)^p = x^p + y^p = t = \alpha^p.$$

This equation can be rewritten as $(x+y-\alpha)^p = 0$. This implies that C' is everywhere non-reduced. Therefore, since the curve C is regular everywhere, we get that for each closed point $x' \in C'$ with image $x \in C$, we have $1 = \dim_{k(x)} T_xC < \dim_{k(x')} T_{x'}C'$.

Theorem 4.2.21. *Let X be an algebraic variety over a perfect field k. Let $d = \dim(X)$. Let $x \in X$ be a closed point. The following are equivalent:*

(1) X is smooth at x;

(2) X is regular at x;

(3) $\dim_{k(x)}(\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} k(x)) = d$.

(4) $\Omega_{X/k,x}$ is a free module of rank d over $\mathcal{O}_{X,x}$;

(5) there exists an open neighbourhood U of $x \in X$ such that $\Omega_{X/k}|_U$ is a locally free $\mathcal{O}_U$-module of rank d.

Proof. We may assume that $X = \operatorname{Spec} A \subset \mathbb{A}^n_k$ is a closed subscheme of $\mathbb{A}^n_k$.

We first prove the equivalence of (1), (2) and (3). Let $\bar{x} \in X_{\bar{k}}$ a closed point lying over $x \in X$. Then by Lemmas 4.2.2, 4.2.13 and 4.2.19, we have

$$\dim(\mathcal{O}_{X,x}) \leq \dim_{k(x)}(T_x X) = \dim_{\bar{k}}(T_{\bar{x}} X_{\bar{k}}) = n - \operatorname{rank}(J_{\bar{x}}) = n - \operatorname{rank}(J_x). \tag{4.4}$$

By Proposition 4.2.15, we have that (1) holds if and only if $\dim(\mathcal{O}_{X,x}) = n - \operatorname{rank}(J_x)$. Thus, in view of (4.4), we see that (1) holds if and only if (2) holds. Moreover, by Corollary 4.2.11, we see that (2) and (3) are equivalent.

Next, observe that (4) and (5) are equivalent in view of Lemma 3.4.7. Moreover, (5) implies (4) implies (3) in a trivial way. We prove that (1) implies (5); then we will be done. Note that if (1) holds, then there exists a neighbourhood U of x in X such that U is smooth over k, see Lemma 4.2.16. Replacing X by U, we may assume that X is smooth over k. We claim that $\Omega_{X/k}$ is locally free of rank d. Indeed, since we have already shown the equivalence of (1) and (3), and since X is smooth at every closed point $z \in X$, we have that $\dim_{k(z)}(\Omega_{X/k,z} \otimes_{\mathcal{O}_{X,z}} k(z)) = d$ for every closed point $z \in X$. Thus, $\Omega_{X/k}$ is locally free of rank d by Exercise 4.2.18. This proves the theorem. $\square$

Example 4.2.22. Let k be a field and let $C = \operatorname{Spec} k[x,y]/(f) \subset \mathbb{A}^2_k$ for some nonzero polynomial $f \in k[x,y]$. Let $R := k[x,y]$ and $A := R/(f)$. By Proposition 4.1.6, we have an exact sequence

$$(f)/(f^2) \longrightarrow A \cdot dx \oplus A \cdot dy \longrightarrow \Omega_{C/k} \longrightarrow 0,$$

where the map

$$(f)/(f^2) \longrightarrow A \cdot dx \oplus A \cdot dy$$

is the map that sends f to $df = f_x dx + f_y dy$ with $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. Therefore:

$$\Omega_{C/k} = \frac{A \cdot dx \oplus A \cdot dy}{f_x dx + f_y dy}. \tag{4.5}$$

In particular, for each $p \in C$, we see that C is smooth at p if and only if $f_x(p) \neq 0$ or $f_y(p) \neq 0$, which happens precisely when $f_x(p)dx + f_y(p)dy \neq 0$, that is, when $\dim_{k(p)}\left(\Omega_{C/k,p} \otimes_{\mathcal{O}_{C,p}} k(p)\right) = 1$ (see (4.5)). This is in accordance with Theorem 4.2.21.

Exercise 4.2.23. Let X be an irreducible scheme of finite type over a field k, all whose irreducible components are of dimension $d \geq 0$. Let $x \in X$ be a closed point. Show that Definitions 3.1.13 and 4.2.14 are equivalent. Prove Lemma 3.1.14.

Chapter 5

Riemann–Roch for curves

5.1 Lecture 26 : Statement of the Riemann–Roch theorem

5.1.1 Divisors on curves

Lemma 5.1.1. *Let C be a regular curve over a field k, with function field K. Let $x \in C$ be a closed point. Then the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{C,x}$ is of the form $\mathfrak{m}_x = (t_x)$ for some $t_x \in \mathcal{O}_{C,x}$. Consequently, having fixed such a generator t_x for $\mathfrak{m}_x$, for each $f \in K^*$ there are unique $\alpha \in \mathcal{O}^*_{C,x}$ and $n \in \mathbb{Z}$ such that $f = \alpha t_x^n$. This defines a valuation*

$$v_x \colon K \longrightarrow \mathbb{Z} \cup \{\infty\}, \qquad \textit{with the property that} \qquad v_x(t_x^n) = n.$$

Moreover, we have $\mathcal{O}_{C,x} = \{f \in K \mid v_x(f) \geq 0\}$.

Proof. See Theorem 3.1.21. $\square$

Lemma 5.1.2. *Let C be a curve over a field k. Any closed subset $Z \subset C$ is either of the form $Z = C$ or of the form $Z = \{x_1, \ldots, x_n\}$ for closed points $x_i \in C$. If a point $x \in C$ is not closed, then $\overline{\{x\}} = C$, i.e. in that case, $x = \eta$ is the generic point of C.*

Proof. If $Z \subsetneq C$ is a closed subset of C, then each irreducible component $W \subset Z$ of Z has dimension $\dim(W) < \dim(C) = 1$. Thus W must be a point. If $x \in C$ is not closed, then $\{x\} \subsetneq \overline{\{x\}}$ which implies that $\overline{\{x\}}$ is an irreducible closed subset of dimension > 0. It must therefore equal C. $\square$

Let C be a regular curve over a field k. Let

$$D = \sum_{x \in C \text{ closed}} n_x \cdot x$$

be a Weil divisor on C. By Proposition 3.1.11, each residue field extension $k(x)$ is a finite field extension of k. Its degree is denoted by $[k(x) \colon k]$.

Definition 5.1.3. The *degree* of the Weil divisor D is the integer $\sum n_x \cdot [k(x) \colon k]$.

Continue to consider the Weil divisor $D = \sum n_x x$. We aim to give a Cartier divisor $\mathcal{D}$ with the property that $\pi(\mathcal{D}) = D$, with respect to the isomorphism $\pi\colon \mathrm{CaDiv}(C) \xrightarrow{\sim} \mathrm{Div}(C)$, see Proposition 3.4.8. To do this, let $x \in C$ be a closed point. Let $U_x \subset C$ be an affine open neighbourhood disjoint of all the $y \in \mathrm{Supp}(D)$ with $y \neq x$. Then define $g_x = t_x^{n_x} \in K^*$. This gives a Cartier divisor $\mathcal{D}$ defined by the Cartier datum $\{(U_x, g_x)\}$ indexed by the closed points of C (remark that $\cup_{x \text{ closed}} U_x = C$ by Lemma 5.1.2). Moreover, we have $\pi(\mathcal{D}) = D$ (verify this!). In particular:

$$\deg(D) = \sum_{x \in C \text{ closed}} [k(x)\colon k] \cdot v_x(g_x).$$

Lemma 5.1.4. *Let C be a smooth curve over a field k and let $x \in C$ be a closed point. Then*

$$\Omega_{k(C)/k} = \Omega_{C/k,x} \otimes_{\mathcal{O}_{C,x}} k(C), \tag{5.1}$$

and this is a $k(C)$-vector space of dimension one.

Proof. Indeed, since the $\mathcal{O}_C$-module $\Omega_{C/k}$ is locally free of rank one (see Theorem 4.2.21), we get that for any point $x \in C$, we have that $\Omega_{C/k,x}$ is an $\mathcal{O}_{C,x}$-module free of rank one. In particular, $\Omega_{k(C)/k} = \Omega_{C/k,\eta}$ is an $\mathcal{O}_{C,\eta} = k(C)$-vector space of dimension one. If $x \in C$ is a closed point, then $\Omega_{C/k,x}$ is a free $\mathcal{O}_{C,x}$-module of rank one, hence $\Omega_{C/k,x} \otimes_{\mathcal{O}_{C,x}} k(C)$ is a $k(C)$-vector space of dimension one. The maps

$$k \longrightarrow \mathcal{O}_{C,x} \longrightarrow k(C)$$

induce a morphism of one-dimensional $k(C)$-vector spaces

$$\varphi\colon \Omega_{C/k,x} \otimes_{\mathcal{O}_{C,x}} k(C) = \Omega_{\mathcal{O}_{C,x}/k} \otimes_{\mathcal{O}_{C,x}} k(C) \longrightarrow \Omega_{k(C)/k}.$$

To prove (5.1), we need to show that φ is not the zero map. For this, let $\mathfrak{m} \subset \mathcal{O}_{C,x}$ be the maximal ideal; let $t \in \mathcal{O}_{C,x}$ so that $\mathfrak{m} = (t)$. Since $t \in \mathcal{O}_{C,x} \subset k(C)$, we get elements $dt \in \Omega_{C/k,x}$ and $dt \in \Omega_{k(C)/k}$. As $\varphi(dt \otimes 1) = dt$, the map φ is non-trivial. □

Definition 5.1.5. Let C be a smooth curve over a field k. Let $\omega \in \Omega_{k(C)/k} - \{0\}$. We define $\mathrm{div}(\omega) \in \mathrm{Div}(C)$ as follows. For each closed point $x \in C$, choose a generator η_x for $\Omega_{C/k,x}$ and write $\omega = g_x \cdot \eta_x$ for some $g_x \in k(C)^*$ (see Lemma 5.1.4). Then

$$\mathrm{div}(\omega) := \sum_{x \in C \text{ closed}} v_x(g_x) \cdot x.$$

Lemma 5.1.6. *Let C be a smooth curve over a field k. Fix $\omega \in \Omega_{k(C)/k} - \{0\}$.*

(1) The element $\mathrm{div}(\omega)$ as defined above does not depend on the choice of the generator $\eta_x \in \Omega_{C/k,x}$ for each closed $x \in C$.

(2) If $\omega' = \lambda \cdot \omega \in \Omega_{k(C)/k}$ with $\lambda \in k(C)^*$, then $\mathrm{div}(\omega') = \mathrm{div}(\lambda) + \mathrm{div}(\omega)$, hence $\mathrm{div}(\omega')$ and $\mathrm{div}(\omega)$ are linearly equivalent.

Proof. Exercise. $\square$

Definition 5.1.7. Let C be a smooth curve over a field k. We define the *canonical divisor class* of C as the Weil divisor class

$$K_C := [\mathrm{div}(\omega)] \in \mathrm{Cl}(C)$$

where ω is any element of $\Omega_{k(C)/k} - \{0\}$. Note that, by Lemma 5.1.6, the canonical divisor class $K_C \in \mathrm{Cl}(C)$ does not depend on the choice of ω.

Proposition 5.1.8. *Let C be a smooth curve over a field k. Then we have a canonical isomorphism of line bundles $\mathcal{O}_C(K_C) \cong \Omega_C$.*

Proof. Exercise. $\square$

Example 5.1.9. Let k be a field. Consider the projective line $\mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$ over k. We claim that

$$\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-2). \tag{5.2}$$

Recall that $\mathrm{Pic}(\mathbb{P}^1_k) = \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^1_k}(1)$, see Exercise 2.3.11. Therefore, in view of the isomorphism $\mathcal{O}_{\mathbb{P}^1_k}(K_{\mathbb{P}^1_k}) \cong \Omega_{\mathbb{P}^1_k/k}$ (see Proposition 5.1.8), to prove (5.2) it suffices to show that $\deg(K_{\mathbb{P}^1_k}) = -2$. Consider the open subscheme $U_0 \subset \mathbb{P}^1_k$ with

$$U_0 = D_+(x_0) = \mathrm{Spec}\ k[x_0, x_1]_{(x_0)} \cong \mathrm{Spec}\ k[t],$$

see Proposition 2.2.1. This gives a rational differential

$$\omega = dt \in \Omega_{k(\mathbb{P}^1_k)/k}$$

which is non-zero. On $U_1 = \mathrm{Spec}\ k[t^{-1}]$, we can write $u = t^{-1}$, and have:

$$dt = d(u^{-1}) = -u^{-2} du.$$

Thus $\mathrm{div}(\omega) = -2 \cdot (0\colon 1)$. In particular, $\deg(\mathrm{div}(\omega)) = -2$, proving (5.2).

To construct a *canonical* isomorphism $\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-2)$, we use Theorem 4.1.10 which gives a canonical exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^1_k/k} \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-1)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow 0.$$

Consider then the composition

$$\Omega_{\mathbb{P}^1_k/k} \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1_k}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-1) \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \mathcal{O}_{\mathbb{P}^1_k}(-1) \cong \mathcal{O}_{\mathbb{P}^1_k}(-2), \tag{5.3}$$

in which the last map is the isomorphism of item (4) of Proposition 1.1.15. It remains to verify that (5.3) is an isomorphism, which we leave as an exercise for the reader.

5.1.2 Genus of a curve and Euler characteristic of a sheaf

Let C be a projective curve over a field k. Recall (cf. Definition 2.4.9), that the *genus* of C is the integer $g(C) = \dim_k \mathrm{H}^1(C, \mathcal{O}_C)$. This is an integer by Theorem 2.3.3.

Examples 5.1.10. (1) We have $g(\mathbb{P}^1_k) = 0$ as $\mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = 0$ by Corollary 2.3.2.

(2) Let $C \subset \mathbb{P}^2_k$ be a plane curve of degree $d > 0$. Then $g(C) = (d-1)(d-2)/2$, see Theorem 2.4.15.

Definition 5.1.11. Let X be a projective variety over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. For $i \in \mathbb{Z}_{\geq 0}$, we define

$$h^i(X, \mathcal{F}) \coloneqq \dim_k \mathrm{H}^i(X, \mathcal{F}), \qquad \chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathcal{F}).$$

Remark that $h^i(X, \mathcal{F}) \in \mathbb{Z}_{\geq 0}$ and $\chi(X, \mathcal{F}) \in \mathbb{Z}_{\geq 0}$ by Theorems 2.3.3 and 2.1.15. The integer $\chi(X, \mathcal{F})$ is called the *Euler characteristic* of the coherent sheaf $\mathcal{F}$.

Lemma 5.1.12. *Let X be a projective variety over a field k. Consider a short exact sequence*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of coherent sheaves on X. Then $\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$.

Proof. Taking cohomology gives a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{F}_1) \longrightarrow \mathrm{H}^0(X, \mathcal{F}_2) \longrightarrow \cdots \longrightarrow \mathrm{H}^n(X, \mathcal{F}_3) \longrightarrow 0.$$

The result follows then from Lemma 2.4.14. □

5.1.3 Riemann–Roch and Serre duality: statement of the theorems

We come to the statements of the Riemann–Roch theorem and the Serre duality theorem for curves.

Theorem 5.1.13 (Riemann–Roch)**.** *Let C be a smooth projective curve over a field k. Let g be the genus of C. Then for any Weil divisor $D \in \mathrm{Div}(C)$, we have:*

$$\chi(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = \deg(D) + 1 - g. \qquad (5.4)$$

Theorem 5.1.14 (Serre duality for curves)**.** *Let C be a smooth projective curve over a field k. Let $\mathcal{F}$ be a finite locally free sheaf on C. Then there are canonical isomorphisms*

$$\mathrm{H}^0(C, \mathcal{F})^\vee = \mathrm{H}^1(C, \mathcal{F}^\vee \otimes_{\mathcal{O}_C} \Omega_{C/k}), \qquad (5.5)$$

$$\mathrm{H}^1(C, \mathcal{F}) = \mathrm{H}^0(C, \mathcal{F}^\vee \otimes_{\mathcal{O}_C} \Omega_{C/k})^\vee. \qquad (5.6)$$

In particular, if $D \in \mathrm{Div}(C)$ is a Weil divisor on C, then there is a canonical isomorphism of k-vector spaces $\mathrm{H}^1(C, \mathcal{O}_C(D)) = \mathrm{H}^0(C, \mathcal{O}_C(K_C - D))^\vee$.

Remark 5.1.15. Observe that (5.6) follows from (5.5). Namely, the canonical morphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$ that sends a local section s to the morphism of sheaves $\mathcal{F}^\vee \to \mathcal{O}_C$ defined on local sections as $(f \mapsto f(s))$, is an isomorphism. Thus, by (5.5), we have

$$\mathrm{H}^1(C, \mathcal{F}) = \mathrm{H}^1\left(C, (\mathcal{F}^\vee \otimes_{\mathcal{O}_C} \Omega_{C/k})^\vee \otimes_{\mathcal{O}_C} \Omega_{C/k}\right) = \mathrm{H}^0(C, \mathcal{F}^\vee \otimes_{\mathcal{O}_C} \Omega_{C/k})^\vee.$$

Corollary 5.1.16. *Let C be a smooth projective curve over a field k. Let g be the genus of C. Then $g = \dim_k \mathrm{H}^0(C, \Omega_C)$.*

Proof. Indeed, we have $g(C) = h^1(C, \mathcal{O}_C) = h^0(C, \Omega_C)$ by Theorem 5.1.14. □

As a corollary of Theorems 5.1.13 and 5.1.14, we obtain:

Theorem 5.1.17 (Riemann–Roch)**.** *Let C be a smooth projective curve of genus g over a field k. Let $D \in \mathrm{Div}(C)$ be a Weil divisor on C. Then*

$$h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(K_C - D)) = \deg(D) + 1 - g.$$

Corollary 5.1.18. *Let C be a smooth projective curve of genus g over a field k. Consider the canonical divisor class $K_C \in \mathrm{Cl}(C)$. Then $\deg(K_C) = 2g - 2$.*

Proof. We have

$$h^0(C, \Omega_C) - h^0(C, \mathcal{O}_C) = h^0(C, \mathcal{O}_C(K_C)) - h^0(C, \mathcal{O}_C(K_C - K_C)) = \deg(K_C) + 1 - g,$$

where the second equality holds by Theorem 5.1.17. As $h^0(C, \mathcal{O}_C) = 1$ (see Example 2.3.4), we get:

$$\deg(K_C) = h^0(C, \Omega_C) + g - 2.$$

By Corollary 5.1.16, we have $h^0(C, \Omega_C) = g$, thus $\deg(K_C) = 2g - 2$ as desired. □

Exercise 5.1.19. Let D be a Weil divisor on a smooth projective curve C over a field k. Assume that $\deg(D) < 0$. Show that $\mathrm{H}^0(C, \mathcal{O}_C(D)) = 0$.

Corollary 5.1.20. *Let C be a smooth projective curve of genus g over a field k. Let $D \in \mathrm{Div}(C)$ be a Weil divisor with $\deg(D) > 2g - 2$. Then $\mathrm{H}^1(C, \mathcal{O}_C(D)) = 0$ and*

$$h^0(C, \mathcal{O}_C(D)) = \deg(D) + 1 - g. \tag{5.7}$$

Proof. Indeed, as $\deg(D) > 2g-2$, we have that $\deg(K_C - D) = \deg(K_C) - \deg(D) < 0$, see Corollary 5.1.18. Therefore, $h^1(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(K_C - D)) = 0$, see Theorem 5.1.14 and Exercise 5.1.19. Thus, (5.7) follows from Theorem 5.1.13. □

5.2 Lecture 27 : Proof of the Riemann–Roch theorem

5.2.1 Morphisms between a variety and a curve

Lemma 5.2.1. *Let X be a projective variety over a field k. Let R be a k-algebra which is a discrete valuation ring, with fraction field K. Then for any morphism of k-schemes $f\colon \mathrm{Spec}\ K \to X$ there exists a unique morphism of k-schemes $g\colon \mathrm{Spec}\ R \to X$ such that $f = g \circ \varphi$, where φ is the canonical morphism* $\mathrm{Spec}\ K \to \mathrm{Spec}\ R$.

Proof. First assume that the lemma is true for projective space of any dimension over k. Then let $X \subset \mathbb{P}^n_k$ be a closed embedding into $\mathbb{P}^n_k$ for some $n \geq 0$. Then since the lemma holds for $\mathbb{P}^n_k$, we get a morphism of k-schemes $\mathrm{Spec}\ R \to \mathbb{P}^n_k$ fitting into the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\ R \arrow[r] & \mathbb{P}^n_k \\ \mathrm{Spec}\ K \arrow[r] \arrow[u] & X. \arrow[u] \end{tikzcd}$$

Let $Z \subset \mathbb{P}^n_k$ be the closure of the image of the morphism $\mathrm{Spec}\ R \to \mathbb{P}^n_k$. Note that $\dim(Z) \in \{0,1\}$, and that $Z \cap X \neq \emptyset$. We claim that $Z \subset X$. Otherwise,

$$\emptyset \neq X \cap Z \subsetneq Z,$$

which implies that $\dim(Z) = 1$ (since Z is connected), and that $X \cap Z \subsetneq Z$ is a closed subset of Z of dimension zero, whereas we have a factorization

$$\varphi\colon \mathrm{Spec}\ K \longrightarrow X \cap Z \hookrightarrow Z$$

and the image of φ is dense in Z, as we have:

$$\overline{\varphi(\mathrm{Spec}\ K)} = \overline{\varphi(\overline{\mathrm{Spec}\ K})} = \overline{\varphi(\mathrm{Spec}\ R)} = Z.$$

This contradiction shows that indeed, $Z \subset X$. This implies that the morphism $\mathrm{Spec}\ R \to \mathbb{P}^n_k$ factors as $\mathrm{Spec}\ R \to Z \subset X \subset \mathbb{P}^n_k$, proving what we want.

It remains to prove the lemma in case $X = \mathbb{P}^n_k$. Let $\mathfrak{m} \subset R$ be the maximal ideal of R, so that $R = (t)$ for some $t \in R$. Note that the morphism $\mathrm{Spec}\ K \to \mathbb{P}^n_k$ corresponds to the class

$$(s_0\colon \cdots : s_n) \in (K^{n+1} - \{0\})_{/\sim}$$

of some $(n+1)$-tuple of elements of K, not all zero, see Example 1.2.21. Write $s_i = \alpha_i \cdot t^{n_i}$ for unique $\alpha_i \in R$ (with $\alpha_i = 0$ or $\alpha_i \in R^*$), and $n_i \in \mathbb{Z}$. Let $I \subset \{0, \dots, n\}$ with $\alpha_i \neq 0$ if and only if $i \in I$. Let m be the minimum of the n_i with $n_i \in I$. Then

$$t^{-m} \cdot t^{n_i} = t^{n_i - m},$$

and $n_i - m \geq m - m = 0$ for each $i \in I$. Hence $t^{-m} \cdot t^{n_i} \in R$ for each $i \in I$. Moreover, there exists $i_0 \in I$ such that $m = n_{i_0}$. Then by Example 1.2.22, the $(n+1)$-tuple

$$t^{-m} \cdot (s_0, \dots, s_n) = (\alpha_0 t^{n_0 - m}, \dots, \alpha_n t^{n_n - m}) \in (R^{n+1} - \{0\})$$

gives rise to a unique morphism

$$\operatorname{Spec} R \longrightarrow \mathbb{P}^n_k$$

whose composition with $\operatorname{Spec} K \to \operatorname{Spec} R$ gives the original map $\operatorname{Spec} K \to \mathbb{P}^n_k$. $\square$

Lemma 5.2.2. *Let X and S be schemes. Let $f, g\colon X \to S$ be two S-scheme morphisms that agree on U, a dense open subset of X. If X is reduced and S separated, then $f = g$.*

Proof. Exercise. $\square$

Proposition 5.2.3. *Let X and C be smooth projective curves over a field k. Let $f\colon X \to C$ be a morphism of schemes over k. Then the following assertions are true.*

(1) Either f is surjective, of f is constant.

(2) If f is surjective, then the fibres of f are finite.

Proof. (1). Consider the subset $f(X) \subset C$. We get a closed connected subset $\overline{f(X)} \subset C$; this subset is either C or a single point (see Lemma 5.1.2). We assume $\overline{f(X)}$ is not a point, so that $\overline{f(X)} = C$. We then need to show that f is surjective. As $\overline{f(X)} = C$, the generic point of C is in the image of f, so we need to prove that for any closed point $x \in C$ there exists a closed point $z \in X$ such that $f(z) = x$.

For this, let $x \in C$ be a closed point. Let $K = \operatorname{Frac}(\mathcal{O}_{C,x})$ be the function field of C, which is also the fraction field of the discrete valuation ring $\mathcal{O}_{C,x}$. Let L be the function field of the curve X. Let R be the integral closure of $\mathcal{O}_{C,x}$ in L. Then R is a discrete valuation ring with fraction field L, and we get a commutative diagram

$$\begin{tikzcd}
 & X \arrow[r] & C \\
 & \operatorname{Spec} R \arrow[u, dotted] \arrow[r] & \operatorname{Spec} \mathcal{O}_{C,x} \arrow[u] \\
\operatorname{Spec} L \arrow[uur, bend left] \arrow[ur] \arrow[rr] & & \operatorname{Spec} K. \arrow[u]
\end{tikzcd} \tag{5.8}$$

By Lemma 5.8, there is a morphism $\operatorname{Spec} R \to X$ that extends the map $\operatorname{Spec} L \to X$. We claim that it makes the square on the top right of (5.8) commute. Indeed, the two compositions $\operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{C,x} \to C$ and $\operatorname{Spec} R \to X \to C$ yield two morphisms $\operatorname{Spec} R \to C$ that agree on the dense open subset $\operatorname{Spec} L \subset \operatorname{Spec} R$ (cf. Lemma 3.1.12). By Lemma 5.2.2, these morphisms $\operatorname{Spec} R \to C$ must then be the same. Now let $y \in \operatorname{Spec} R$ be the closed point of $\operatorname{Spec} R$, and let $z \in X$ be the image of y under $\operatorname{Spec} R \to X$. Then $z \in X$ is a closed point such that $f(z) = x$.

(2). Assume $f\colon X \to C$ is surjective. Let $x \in C$ be a point. If $x = \xi$ is the generic point of C, then $f^{-1}(x) = \eta$ is the generic point of X (indeed, this follows the fact that f maps closed points to closed points, and that a point on a curve is closed if and only if it is not the generic point; cf. Lemma 5.1.2). If $x \in C$ is a closed point of C, then $f^{-1}(x) \subset X$ is a closed subset of X strictly contained in X, and therefore, by Lemma 5.1.2, $f^{-1}(x)$ consists of finitely many closed points of X. $\square$

Proposition 5.2.4. *Let X and C be varieties over a field k, with X projective and C a curve. Let $U \subset C$ be a non-empty open subset of C, and let $f^0 : U \to X$ be a morphism of k-schemes. Then f^0 admits a unique extension $f : C \to X$.*

Proof. We may assume that $X = \mathbb{P}^n_k$ for some $n \geq 0$ (verify this). If $U \neq C$, then $C - U =: Z$ consists of finitely many closed points of C (see Lemma 5.1.2). To prove the proposition, we may assume Z is a single closed point $x \in C$. Thus $U = C - \{x\}$.

Let K be the function field of C, so that $K = \mathrm{Frac}(\mathcal{O}_{C,x})$. By Lemma 5.2.1, there exists a unique morphism

$$\mathrm{Spec}\ \mathcal{O}_{C,x} \longrightarrow \mathbb{P}^n_k$$

that extends the composition $\mathrm{Spec}\ L \to U \to \mathbb{P}^n_k$. Thus we get $n+1$ sections $(u_0, \dots, u_n) \in \mathcal{O}^{n+1}_{C,x} - \{0\}$ that do not vanish at the maximal ideal of $\mathcal{O}_{C,x}$ (see Example 1.2.22). By Theorem 1.2.18, the morphism $U \to \mathbb{P}^n_k$ corresponds to a line bundle $L = \mathcal{O}_U(D)$ attached to some Weil divisor D on U, together with $n+1$ sections $s_0, \dots, s_n \in \Gamma(U, \mathcal{O}_U(D))$ that globally generate L. If $D = \sum_i n_i x_i \in \mathrm{Div}(U)$, define $\bar{D} \in \mathrm{Div}(C)$ as the Weil divisor

$$\bar{D} \coloneqq \sum_i n_i x_i \in \mathrm{Div}(C).$$

We get a line bundle $\bar{L} \coloneqq \mathcal{O}_C(\bar{D})$ on C. Let $x \in V \subset C$ be an open neighbourhood such that $\bar{L}|_V \cong \mathcal{O}_V$. By shrinking V around x if necessary, we may assume that the elements $u_i \in \mathcal{O}^{n+1}_{C,x}$ extend to sections of $\bar{L}$ over V which, up to multiplication, agree with the sections s_i over $V - \{x\}$. By further shrinking V around x if necessary, we may assume that the sections $u_0, \dots, u_n \in \Gamma(V, \bar{L})$ globally generate $\bar{L}$. By Theorem 1.2.18, this gives a unique morphism

$$V \longrightarrow \mathbb{P}^n_k$$

that extends the composition $V - \{x\} \subset U \to \mathbb{P}^n_k$. The proposition follows. □

5.2.2 Rational functions on curves

Proposition 5.2.5. *Let X and Y be projective schemes over a ring A. Assume $f : X \to Y$ is a morphism of schemes over A with finite fibers. Then for each affine open subscheme $U \subset Y$, the subscheme $f^{-1}(U) \subset X$ is affine.*

Proof. We do not prove this here. □

Let X be a variety over a field k, with generic point $\eta \in X$. Then $k(X) = \mathcal{O}_{X,\eta}$ is a field, and called the function field of X. For a non-empty affine open $U = \mathrm{Spec}\ A \subset X$, we have $k(X) = \mathrm{Frac}(A)$. Indeed, to prove this, we may assume $X = \mathrm{Spec}\ A$ is affine, in which case the generic point corresponds to the zero ideal of A.

Lemma 5.2.6. *Let X be a variety over a field k.*

(1) *For any open $U \subset X$, there is a natural bijection $\Gamma(U, \mathcal{O}_X) = \mathrm{Hom}_{\mathsf{Sch}/k}(U, \mathbb{A}^1_k)$.*

(2) *There exists a natural bijection between $k(X)$ and the set of equivalence classes of tuples (U, f) where $U \subset X$ is a non-empty open and*

$$f \colon U \longrightarrow \mathbb{A}^1_k$$

is a morphism from U to the affine line over k, and where $(U, f) \sim (V, g)$ if the maps f and g agree on the open subset $U \cap V \subset X$.

Proof. As for (1): remark that

$$\mathrm{Hom}_{\mathsf{Sch}/k}(U, \mathbb{A}^1_k) = \mathrm{Hom}_{\mathsf{Sch}/k}(U, \mathrm{Spec}\ k[t]) = \mathrm{Hom}_{k\text{-}\mathsf{Alg}}(k[t], \mathcal{O}_X(U)) = \mathcal{O}_X(U).$$

As for (2): this follows readily from (1) and the definition of $k(X)$. $\square$

In combination with the results of Section 5.2.1, the following lemma yields:

Proposition 5.2.7. *Let C be a smooth projective curve over a field k. Let $\mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$ with $\infty = (0\colon 1) \in \mathbb{P}^1_k(k)$. Then $\mathbb{P}^1_k - \{\infty\} \cong \mathbb{A}^1_k$. We have:*

(1) *There is a natural bijection*

$$k(C) = \left\{ f \in \mathrm{Hom}_{\mathsf{Sch}/k}(C, \mathbb{P}^1_k) \mid f(C) \neq \{\infty\} \right\}.$$

(2) *In particular, C admits a non-constant morphism $f \colon C \to \mathbb{P}^1_k$.*

(3) *Let $f \colon C \to \mathbb{P}^1_k$ be non-constant. Then f is surjective with finite fibers. Moreover, for any affine open $U \subset \mathbb{P}^1_k$, the inverse image $f^{-1}(U) \subset C$ is affine.*

Proof. (1). By Lemma 5.2.6, any element of $k(C)$ corresponds to the equivalence class of a morphism $f^0 \colon U \to \mathbb{A}^1_k$ defined on a non-empty open $U \subset C$. By Proposition 5.2.4, the composition

$$f^0 \colon U \longrightarrow \mathbb{A}^1_k \longrightarrow \mathbb{P}^1_k$$

extends to a unique morphism

$$f \colon C \longrightarrow \mathbb{P}^1_k.$$

Note that $f(C) \neq \{\infty\}$. This construction yields the desired bijection.

(2). The subfield $k \subset k(C)$ corresponds to the set of constant maps $C \to \mathbb{A}^1_k \subset \mathbb{P}^1_k$. For a non-empty affine open $U \subset C$, we have $k(C) = \mathrm{Frac}(\mathcal{O}_C(U))$. Thus $k \subsetneq k(C)$ is strictly contained in $k(C)$, so that there exists a non-constant map $f \colon C \to \mathbb{P}^1_k$.

(3). Let $f \colon C \to \mathbb{P}^1_k$ be non-constant. Then f is surjective with finite fibers by Proposition 5.2.3. Since C and $\mathbb{P}^1_k$ are projective over k, we get that for each affine open $U \subset \mathbb{P}^1_k$, the inverse image $f^{-1}(U)$ is an affine scheme, see Proposition 5.2.5. $\square$

5.2.3 Proof of Serre–Duality for the projective line

We first prove Serre–Duality for the curve $C = \mathbb{P}^1_k$. For this, we use the following theorem as a blackbox.

Theorem 5.2.8. *Let k be a field. Let $\mathcal{F}$ be a locally free sheaf of finite rank $n \in \mathbb{Z}_{\geq 1}$ on the projective line $\mathbb{P}^1_k$. Then there are integers $a_1, \dots, a_n \in \mathbb{Z}$ such that*

$$\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_n).$$

Proof. When $n = 1$, this is Exercise 2.3.11. We do not prove the general case here. □

We can then show:

Lemma 5.2.9 (Serre duality on $\mathbb{P}^1_k$)**.** *Let $\mathcal{F}$ be a finite locally free sheaf on $\mathbb{P}^1_k$. Then there is a natural isomorphism of k-vector spaces*

$$\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{F})^\vee = \mathrm{H}^1(\mathbb{P}^1_k, \mathcal{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \Omega_{\mathbb{P}^1_k/k}). \tag{5.9}$$

Proof. By Theorem 5.2.8, we have $\mathcal{F} \cong \oplus_i \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ for some integers $a_1, \dots, a_n \in \mathbb{Z}$. In particular, as taking cohomology of a sheaf commutes with direct sums, it suffices to prove (5.9) in the case where $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1_k}(a)$ for some $a \in \mathbb{Z}$. Notice that $\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-2)$ by Example 5.1.9. Hence

$$\mathcal{F}^\vee \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-a-2),$$

see item (1) of Exercise 2.3.11. Therefore, we need to provide a natural isomorphism

$$\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a))^\vee = \mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-a-2)). \tag{5.10}$$

We have $\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(a)) = k[x_0, x_1]_a$ and

$$\mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-a-2)) = \left((x_0x_1)^{-1} \cdot k[x_0^{-1}, x_1^{-1}]\right)_{-a-2},$$

see Theorem 2.3.1. Now we have a pairing

$$\begin{aligned} k[x_0, x_1]_a \times \left((x_0x_1)^{-1} \cdot k[x_0^{-1}, x_1^{-1}]\right)_{-a-2} &\longrightarrow k, \\ (f, g) &\mapsto f \cdot x_0^2 x_1^2 \cdot g. \end{aligned}$$

This pairing is perfect, providing the desired isomorphism (5.10). □

5.2.4 Preliminary results for Serre duality for curves

In this section, we gather several results that we need in the proof of Theorem 5.1.14.

Proposition 5.2.10. *Let C be a smooth projective curve over a field k. Let $\pi \colon C \to \mathbb{P}^1_k$ be a non-constant morphism. Let $\mathcal{F}$ be a finite locally free $\mathcal{O}_C$-module. Then the $\mathcal{O}_{\mathbb{P}^1_k}$-module $\pi_*\mathcal{F}$ is finite locally free, and we have an isomorphism of $\mathcal{O}_{\mathbb{P}^1_k}$-modules*

$$\pi_*\left(\mathscr{H}om_{\mathcal{O}_C}(\mathcal{F}, \Omega_{C/k})\right) \cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^1_k}}(\pi_*\mathcal{F}, \Omega_{\mathbb{P}^1_k/k}).$$

Proof. We first prove that $\pi_*\mathcal{F}$ is finite locally free. Since $\mathcal{F}$ is finite locally free, we have that for each affine open $W \subset C$, we have that $\mathcal{F}(W)$ is a flat $\mathcal{O}_C(W)$-module (verify this!).

Let $V \subset \mathbb{P}^1_k$ be any affine open. Then $U := \pi^{-1}(V)$ is affine by Proposition 5.2.5. Let $A = \mathcal{O}_{\mathbb{P}^1_k}(V)$ and $B = \mathcal{O}_C(U)$. The resulting ring map $A \to B$ is flat (verify this!). Moreover, by the above, the module $(\pi_*\mathcal{F})(V) = \mathcal{F}(U)$ is flat over B. Therefore, $\mathcal{F}(U)$ is flat over A.

Hence we see that for each affine open $V \subset \mathbb{P}^1_k$, the $\mathcal{O}_{\mathbb{P}^1_k}(V)$-module $(\pi_*\mathcal{F})(V)$ is flat. In particular, for each $x \in \mathbb{P}^1_k$, we get that $(\pi_*\mathcal{F})_x$ is a flat $\mathcal{O}_{\mathbb{P}^1_k,x}$-module of finite type. Since $\mathcal{O}_{\mathbb{P}^1_k,x}$ is a discrete valuation ring, any flat finite type module over it is finite free. This proves that $(\pi_*\mathcal{F})_x$ is a finite free $\mathcal{O}_{\mathbb{P}^1_k,x}$-module for all $x \in \mathbb{P}^1_k$. Therefore $\pi_*\mathcal{F}$ is finite locally free by Lemma 3.4.7.

Next, we prove that there exists a canonical isomorphism

$$\pi_* \left(\mathscr{H}om_{\mathcal{O}_C}(\mathcal{F}, \Omega_{C/k})\right) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^1_k}}(\pi_*\mathcal{F}, \Omega_{\mathbb{P}^1_k/k}). \tag{5.11}$$

To provide the isomorphism (5.11), we define

$$\omega_{\mathbb{P}^1_k} := \mathcal{O}_{\mathbb{P}^1_k}(-2).$$

Note that $\omega_{\mathbb{P}^1_k} \cong \Omega_{\mathbb{P}^1_k/k}$ canonically by Example 5.1.9. We then proceed in two steps:

Step 1 We construct a coherent $\mathcal{O}_C$-module ω_C, together with a canonical isomorphism

$$\pi_* \left(\mathscr{H}om_{\mathcal{O}_C}(\mathcal{F}, \omega_C)\right) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^1_k}}(\pi_*\mathcal{F}, \omega_{\mathbb{P}^1_k}). \tag{5.12}$$

Step 2 We construct a canonical isomorphism $\Omega_{C/k} \cong \omega_C$.

Step 1. The definition of ω_C goes as follows. Consider

$$U_i = D_+(x_i) \subset \mathbb{P}^1_k.$$

We have $U_0 \cong \operatorname{Spec} k[t]$ and $U_1 \cong \operatorname{Spec} k[t^{-1}]$, which are glued along $U_0 \cap U_1 \cong \operatorname{Spec} k[t, t^{-1}]$. Define

$$C_0 := \pi^{-1}(U_0), \qquad C_1 := \pi^{-1}(U_1).$$

Moreover, define

$$A_i := \mathcal{O}_{\mathbb{P}^1_k}(U_i), \qquad B_i := \mathcal{O}_C(C_i) \qquad (i \in \{0, 1\}).$$

Then the $C_i \subset C$ are open subschemes, affine by Proposition 5.2.5, and the restrictions of π gives two morphisms

$$\pi_i \colon C_i \longrightarrow U_i \qquad (i \in \{0, 1\}).$$

Define

$$M_i := \operatorname{Hom}_{A_i}(B_i, \omega_{\mathbb{P}^1_k}(A_i)).$$

Then M_i is a finitely generated B_i-module. Thus $\widetilde{M_i}$ is a coherent $\mathcal{O}_{C_i}$-module. We then glue together $\widetilde{M_0}$ and $\widetilde{M_1}$ to get a coherent sheaf ω_C on C.

We claim that we have an isomorphism as in (5.12). To prove this, we work locally again: define $F_i := \mathcal{F}(C_i)$. Then F_i is a finitely generated B_i-module, and the map

$$\Psi_i \colon \mathrm{Hom}_{B_i}(F_i, \mathrm{Hom}_{A_i}(B_i, \omega_{\mathbb{P}^1_k}(A_i))) \longrightarrow \mathrm{Hom}_{A_i}(F_i, \omega_{\mathbb{P}^1_k}(A_i))$$

defined as

$$\mathrm{Hom}_{B_i}(F_i, \mathrm{Hom}_{A_i}(B_i, \omega_{\mathbb{P}^1_k}(A_i))) \ni \phi \mapsto (\ell \mapsto \phi(\ell)(1)) \in \mathrm{Hom}_{A_i}(F_i, \omega_{\mathbb{P}^1_k}(A_i))$$

is an isomorphism. The maps Ψ_i for $i \in \{0,1\}$ sheafify to an isomorphism (5.12).

Step 2. It remains to construct a canonical isomorphism

$$\Omega_{C/k} \xrightarrow{\sim} \omega_C. \tag{5.13}$$

For this, see [OE15, pages 402 & 403]. □

We proceed with the following lemmas, which we need (together with Proposition 5.2.10) in order to prove Theorem 5.1.14.

Lemma 5.2.11. *Let $f \colon X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_Y$-module. Then there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Proof. Let $A \to B$ be a morphism of rings. Let M be a B-module. Let $n \in \mathbb{Z}_{\geq 1}$ and consider the free A-module A^n. Then there is a natural isomorphism of A-modules

$$M \otimes_B (A^n \otimes_A B) \cong M \otimes_A A^n.$$

The lemma follows from this. □

Lemma 5.2.12. *Let X be a noetherian scheme and let $\mathcal{F}$ and $\mathcal{G}$ be finite locally free $\mathcal{O}_X$-modules. Then we have a canonical isomorphism $\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G} = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.*

Proof. There is indeed a morphism of sheaves $\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ defined as $f' \otimes g \mapsto (f \mapsto f'(f) \cdot g)$. This map is an isomorphism, as can be verified on stalks. □

5.2.5 *Proof of the Serre duality theorem for curves*

Proof of Theorem 5.1.14. We can now prove Theorem 5.1.14. Let C be a smooth projective curve over a field k, and let $\mathcal{F}$ be a finite locally free sheaf on C. In view of Remark 5.1.15, it suffices to prove (5.5). By Lemma 5.2.7, there exists a non-constant morphism

$$\pi \colon C \longrightarrow \mathbb{P}^1_k.$$

By Proposition 5.2.10, we have a canonical isomorphism

$$\pi_* \left(\mathscr{H}om_{\mathcal{O}_C}(\mathcal{F}, \Omega_{C/k})\right) \cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^1_k}}(\pi_*\mathcal{F}, \Omega_{\mathbb{P}^1_k/k}).$$

This yields:

$$\begin{aligned}
\mathrm{H}^0(C,\mathcal{F})^\vee &\cong \mathrm{H}^0(\mathbb{P}^1_k, \pi_*\mathcal{F})^\vee && (5.14)\\
&\cong \mathrm{H}^1\left(\mathbb{P}^1_k, (\pi_*\mathcal{F})^\vee \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \Omega_{\mathbb{P}^1_k}\right) && (5.15)\\
&\cong \mathrm{H}^1\left(\mathbb{P}^1_k, \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^1_k}}(\pi_*\mathcal{F}, \Omega_{\mathbb{P}^1_k})\right) && (5.16)\\
&\cong \mathrm{H}^1\left(\mathbb{P}^1_k, \pi_*\left(\mathscr{H}om_{\mathcal{O}_C}(\mathcal{F}, \Omega_{C/k})\right)\right) && (5.17)\\
&\cong \mathrm{H}^1\left(C, \mathscr{H}om_{\mathcal{O}_C}(\mathcal{F}, \Omega_{C/k})\right) && (5.18)\\
&\cong \mathrm{H}^1\left(C, \mathcal{F}^\vee \otimes_{\mathcal{O}_C} \Omega_{C/k}\right). && (5.19)
\end{aligned}$$

Let us explain the above isomorphisms. The isomorphism (5.14) holds since we have

$$\mathrm{H}^0(\mathbb{P}^1_k, \pi_*\mathcal{F}) = \Gamma(\mathbb{P}^1_k, \pi_*\mathcal{F}) = \Gamma(C,\mathcal{F}) = \mathrm{H}^0(C,\mathcal{F}).$$

Then (5.15) follows from Lemma 5.2.9. The isomorphism (5.16) holds by Lemma 5.2.12. Then (5.17) follows from Proposition 5.2.10. The isomorphism (5.18) follows from Lemma 2.1.17 together with item (3) in Lemma 5.2.7. Finally, (5.19) holds by Lemma 5.2.12 again.

This proves (5.5), and hence we are done. □

5.2.6 *Proof of the first version of Riemann–Roch*

Lemma 5.2.13. *Let C be a smooth projective curve over a field k. Let $D \in \mathrm{Div}(C)$ be a Weil divisor on C. Let $p \in C$ be a closed point. Then:*

$$\chi(C, \mathcal{O}_C(D+p)) = \chi(C, \mathcal{O}_C(D)) + [k(p) \colon k], \quad (5.20)$$

$$\deg(D+p) = \deg(D) + [k(p) \colon k]. \quad (5.21)$$

Proof. Let $\mathcal{I} \subset \mathcal{O}_C$ be the ideal sheaf of the closed subscheme $i \colon \operatorname{Spec} k(p) \hookrightarrow C$ attached to the closed point $p \in C$. By Exercise 3.3.9, we have $\mathcal{I} = \mathcal{O}_C(-p)$ as subsheaves of $\mathcal{O}_C$. This gives an exact sequence

$$0 \longrightarrow \mathcal{O}_C(-p) \longrightarrow \mathcal{O}_C \longrightarrow i_*\mathcal{O}_{\operatorname{Spec} k(p)} \longrightarrow 0.$$

Consider the line bundle $\mathcal{O}_C(D+p)$ on C. Since this $\mathcal{O}_C$-module is invertible, tensoring the above sequence with it gives a sequence which remains exact:

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D+p) \longrightarrow i_*\mathcal{O}_{\operatorname{Spec} k(p)} \longrightarrow 0. \quad (5.22)$$

Here, we used Lemma 5.2.11, which implies that for any line bundle L on C, we have

$$L \otimes_{\mathcal{O}_C} i_*\mathcal{O}_{\operatorname{Spec} k(p)} \cong i_*\left(i^*(L) \otimes_{\mathcal{O}_{\operatorname{Spec} k(p)}} \mathcal{O}_{\operatorname{Spec} k(p)}\right) \cong i_*\mathcal{O}_{\operatorname{Spec} k(p)}.$$

Consider the exact sequence (5.22). By Lemma 5.1.12, we obtain:

$$\chi(C, \mathcal{O}_C(D+p)) = \chi(C, \mathcal{O}_C(D)) + \chi(C, i_*\mathcal{O}_{\operatorname{Spec} k(p)}).$$

Since

$$\begin{aligned}\chi(C, i_*\mathcal{O}_{\operatorname{Spec} k(p)}) &= \dim_k \mathrm{H}^0(C, i_*\mathcal{O}_{\operatorname{Spec} k(p)}) = \dim_k(\mathcal{O}_{\operatorname{Spec} k(p)}(\operatorname{Spec} k(p))) \\ &= \dim_k(k(p)) = [k(p) \colon k],\end{aligned}$$

the equality (5.20) follows.

Consider $p \in C$ as a Weil divisor on C. Then $\deg(p) = [k(p) \colon k]$, so that we have $\deg(D+p) = \deg(D) + \deg(p) = \deg(D) + [k(p) \colon k]$. In particular, (5.21) follows. $\square$

Proof of Theorem 5.1.13. Let C be a smooth projective curve over a field k. Let g be the genus of C, and let $D \in \operatorname{Div}(C)$ be a Weil divisor on C. Write

$$D = \sum_{i=1}^{m} n_i \cdot p_i,$$

for closed points $p_1, \ldots, p_m \in C$. By Lemma 5.2.13, we have

$$\begin{aligned}\chi(C, \mathcal{O}_C(D)) &= \chi(C, \mathcal{O}_C) + \sum_{i=1}^{m} n_i \cdot [k(p_i) \colon k], \\ \deg(D) &= \sum_{i=1}^{m} n_i \cdot [k(p_i) \colon k].\end{aligned}$$

Therefore, to prove (5.4), it suffices to prove that

$$\chi(C, \mathcal{O}_C) = 1 - g.$$

As $\chi(C, \mathcal{O}_C) = h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C)$, and as $h^0(C, \mathcal{O}_C) = 1$ by Example 2.3.4, this amounts to proving that $h^1(C, \mathcal{O}_C) = g$, which holds by Corollary 5.1.16. $\square$

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